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Energy balance climate models and general equilibrium optimal mitigation policies

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ABSTRACT

In a general equilibrium model of the world economy, we develop a two-dimensional energy balance climate model featuring heat diffusion and anthropogenic forcing driven by global fossil fuel use across the sphere of the Earth. This introduces an endogenous location dependent temperature function, driving spatial characteristics, in terms of location dependent damages resulting from local temperature anomalies into the standard climate-economy framework. We solve the social planner's problem and characterize the competitive equilibrium for two polar cases differentiated by the degree of market integration. We define optimal taxes on fossil fuel use and how they may implement the planning solution. Our results suggest that if the implementation of international transfers across latitudes is not possible then optimal taxes are in general spatially non-homogeneous and may be lower at poorer latitudes. The degree of spatial differentiation of optimal taxes depends on heat transportation. By employing the properties of the spatial model, we show by numerical simulations how the impact of thermal transport across latitudes on welfare can be studied.

1. Introduction

The economics of climate change are based on the use of integrated assessment models (IAMs) and focus mainly on the impact of greenhouse gasses (GHGs) emissions on global temperature, as well as the impact from increases of global temperature on economic variables such as output and utility from consumption. IAMs are also used for developing long-term projections regarding climate and economic variables, the cost of climate change, and the formulation of mitigation and climate policies.[1]

The major IAMs which are structured as optimal growth models with a climate component and which are focusing on cost-benefit analysis and policy simulations (e.g. DICE/RICE, MERGE, FUND) simplify the carbon cycle and the climate system considerably, and provide results regarding temperature at a global level. When however the geographical scale is global in terms of temperature and/or damages due to climate change, important aspects of the problem, which are related both to

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[1] See for example DICE/RICE, MERGE, FUND, WITCH.

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climate science and economics are obscured. In particular there are natural mechanisms which induce a spatially non-homogeneous distribution of the surface temperature across the globe. The main drivers of these mechanisms are heat transportation that balances incoming and outgoing radiation, and the differences among the local heat absorbing capacity, the local co-albedo, which is relatively lower in ice covered regions and which changes with time as global warming tends to diminish the ice-cups.

However, these mechanisms are not accounted for by cost-benefit oriented IAMs, and as a result the existing variations in local temperatures are not modeled and therefore they do not affect local damages and their dynamics.[2] This, however, may introduce a serious bias into the result and the policy prescriptions, since it is clear that climate change is going to have impacts with profound regional differentiation across the globe.

From the point of view of climate science these IAMs are zero-dimensional models since they do not include spatial aspects such as heat diffusion. This is not however the case for the one- or two-dimensional energy balance climate models (EBCMs) developed by climate scientists which model heat diffusion across latitudes (one-dimensional) or across latitudes and longitudes (two-dimensional) (see e.g. Budyko, 1969; Sellers, 1969, 1976; North, 1975a,b; North et al., 1981, 1983; Kim and North, 1992; Wu and North, 2007).[3] One-dimensional EBCMs predict a concave temperature distribution across latitudes with the maximum temperature at the equator. This non-uniform temperature distribution is important for understanding the so called temperature anomaly which is the difference between the temperature distribution at a given benchmark period and the current period. Data indicate (Hansen et al., 2010) that since 1880 the anomaly has been relatively higher in high latitude zones, relative to zones around the equator, which suggest spatial non-uniformity in the distribution of temperature over time.

In previous papers, Brock et al. (forthcoming), Brock et al. (2013a), we have explored the properties of the one-dimensional energy balance climate models, showing how they could be coupled to economic growth models in a tractable manner. The analysis revealed new complexities stemming from climate science which also proved to have potential qualitative effects on optimal mitigation policy. The results that could be derived from these models, were however limited in the geopolitical sense, since the spatial analysis remains constrained to interaction across latitudes as opposed interactions across countries or regions.

In the present paper we will take the next natural step following the one-dimensional case, by deriving a two-dimensional model. Here the second dimension will allow for a continuous representation of local temperature anomalies for every latitude and longitude across the surface of the Earth. As with our previous papers, we will be working with a dynamic climate-economy growth model involving both heat diffusion and albedo differentiation across different geographical locations. The solution method is similar to the one-dimensional case but involves finding an alternative orthogonal basis set for the Laplacian of the heat diffusion equation. Due to the added dimensionality this basis set will differ from the Legendre polynomial basis that we used in the one-dimensional models featured in Brock et al. (forthcoming), Brock et al. (2013a), and will now involve an expansion in terms of so called spherical harmonics which are the eigenfunctions of the solution to the two dimensional Laplacian. We believe that this approach which integrates solution methods for two-dimensional spatial climate models, with methods of solving economic models, can help to push the frontier of integrated assessment modeling further by showing how solution methods involving in complex climate models such as Earth system models or general circulation models can be made tractable also when coupled to economic models.[4]

Since EBCMs most likely are new to most economists we have chosen to focus on the basic general equilibrium properties of the derived model under different assumptions regarding international capital markets and their integration. This approach shows of the basic welfare properties of the model and how the fundamental theorems of welfare economics apply within the context of our energy balance climate economy framework. The approach also makes clear how the optimal carbon tax rates should be chosen in order to implement a social planning problem and should thus constitute an important first step for economists working with these types of models. The economic part of the model is an infinite horizon Ramsey-type model allowing for basic heterogeneity among consumers and firms at each respective geographical location on the surface of the Earth identified by its latitude and longitude coordinate. We have chosen to look at two polar cases here. The first two cases concern economies that are either completely open with respect to trade and transfers or completely closed in the sense of being autarkic. In the first case when international trade and transfers are available capital returns and interest rates will be equal across locations. In this case we show that the optimal carbon tax must also be

[2] The DICE/RICE models do not include the spatial transportation of heat, nor the albedo differentials across locations, and perform their analysis in terms of the global mean surface temperature which does not vary across regions during their planning horizons. Nordhauss RICE 2010 divides the world into US, EU, Japan, Russia, Eurasia, China, India, Middle East, Africa, Latin America, other high income, other developing Asia. In the DICE model spatial damages are implicit in the aggregate representation since regional impacts are aggregated to a single measure using a bottom-up approach.

[3] For more on EBCMs see for example Pierrehumbert (2008) (Chapters 3 and 9, especially sections 9.2.5 and 9.2.6 and surrounding material). North et al. (1981) is a very informative review of EBCMs while Wu and North (2007) is a recent paper on EBCMs.

[4] The solution methods for large scale general circulation models are typically divided into *spectral methods* and *finite difference methods*. Spherical harmonics is the usual basis function involved in the spectral method. We could of course had considered other approaches alternative to EBCMs for approximating temperature fields which are based on more complex and computationally costly models, such as pattern scaling (Lopez et al., 2012) or emulation theory (Challenor et al., 2006). Because the purpose of this paper is however to construct the simplest coupled climate economy model with a climate feedback response mechanism in space that responds to changes in the spatiotemporal structure of taxes on fossil fuels, which is still analytically tractable, we considered the EBMCs framework as more appropriate.

uniform or equal across locations. In the second case, of autarkic economies, interest rates and tax rates will generally differ across latitudes. In this case if some countries are rich and some poor and international transfers are assumed to be restricted across latitudes, this will in general imply that optimal carbon taxes will be spatially differentiated. This result that in the absence of international transfers a spatially uniform optimal tax rate is in general not possible was first noted by Chichilnisky and Heal (1994). Our result provides new insights into this issue by characterizing the spatial distribution of fossil fuel taxes and linking the degree of spatial differentiation of optimal fossil fuel taxes to the diffusion of heat across latitudes.[5] Using these two polar cases we are able to reveal aspects of how heat transport across latitudes matters regarding the prediction of spatial distributions and the corresponding temporal evolutions of temperature, damages and optimal mitigation efforts. Our two-dimensional model further allows us to show how heat diffusion across geographical zones impacts on the size of the spatial differentiation of fossil fuel taxes between poor and wealthy regions. In the numerical section of the paper we finally attempt a tentative calibration exercise and show how the autarkic economy can be solved numerically in order to obtain graphical representations of e.g. damage and temperature distributions.

To sum up, the main objective of this paper has been to introduce the economics profession to spatial EBCMs with heat transport as a potentially useful tool for studying the economics of climate change relative to alternative zero dimensional models. By deriving conditions for the spatial distribution for optimal taxes for the two-dimensional coupled climate economic model, we show how the spatial EBCMs can contribute to the current debate regarding how much to mitigate now, whether mitigation policies should be spatially homogeneous or not, and how to derive geographically specific information regarding damages and policy measures.[6]

This paper is structured as follows. In Section 2 we present the two dimensional energy balance climate model which incorporates human impacts on climate resulting from carbon dioxide accumulation due to the use of fossil fuels, that blocks outgoing radiation. In developing the model we follow North et al. (1983) and use his notation.[7] We use solution methods based on spherical harmonics for partial differential equations (PDEs), approximating the solution by a finite set of ordinary differential equations (ODEs). The approach will be used to solve, and numerically approximate location specific temperature and damage functions. Section 3 couples the spatial EBCM with an economic growth model, where a finite stock of fossil fuel is an essential input along with capital and labor. Fossil fuels are extracted by fossil fuel firms which pay taxes on profits and/or taxes per unit of fossil fuel extracted. We solve the model for the social planner and for the competitive equilibrium with taxes. We derive the optimal taxes and their temporal profiles. In Section 4 we use approximate solutions, and simulate the model and in order to derive explicit numerical solutions. Here, we apply an extension related to a paper by Alexeev et al. (2005) which captures a process known as *polar amplification* which is the empirical observation that for a given amount of global warming, local warming tends to be amplified in the polar regions. This shows how our framework can also quite easily incorporate recent developments in the climate community. The last section concludes.

2. An Energy Balance Climate Model with human inputs

In this section we develop a two-dimensional Energy Balance Climate Model with a human input of atmospheric carbon dioxide. The addition of a human input will form the connection to the economic model developed in the following sections. The term “two-dimensional” means that there are two explicit spatial dimensions, latitude and longitude, so that our unified model of the climate and the economy will evolve both in the time and space.[8] We will start off with a fairly general

[5] In Brock et al. (2013a), we also considered a third example for a one-dimensional EBCM. This concerned the case of costly transfers i.e. a transfer economy where transfers are available but come at a cost. In this case we show that under the assumption that the marginal costs of international transfers does not change over time we can have both a spatially differentiated tax rate and equality among interest rates across latitudes. We ignore this in the current paper.

[6] Another issue that can be addressed by latitude dependent climate models is damage reservoirs. Damage reservoirs in the contest of climate change can be regarded as sources of climate damages which will eventually cease to exist when the source of the damages is depleted. Damage reservoirs are latitude dependent and ice lines and permafrost can be regarded as such reservoirs.

As the ice lines move closer to the poles, due to climate change, we might expect that marginal damages from this moving will be large at first and then diminish as the ice line approaches the Poles. When there will be no ice left on the Poles this damage reservoir would have been exhausted. The presence of an endogenous ice line in the EBCM allows us to model these types of damages explicitly given the relevant information.

Permafrost is soil at or below the freezing point of water for two or more years. The permafrost feedback suggests that permafrost carbon emissions could affect long-term projections of future temperature change. Studies indicate that up to 22% of permafrost could be thawed already by 2100. Once unlocked under strong warming, thawing and decomposition of permafrost can release amounts of carbon until 2300 comparable to the historical anthropogenic emissions up to 2000 (approximately 440 GTC) (Schneider et al., 2011). Alexeev et al. (2005), Graversen et al. (2008) and Alexeev and Jackson (2012) discuss mechanisms that cause polar amplification. Polar amplification could lead to increased risk of permafrost melt. A very recent expert assessment of the vulnerability of permafrost carbon to climate change by Schuur et al. (2013) shows that this risk is believed to be substantial among experts.

The modeling of damage reservoirs is beyond the scope of the present paper but we have addressed this within the context of a one-dimensional in a separate paper of ours (see Brock et al., forthcoming). Our findings are that the introduction of damage reservoirs into the climate-economy framework may give rise to multiple steady states and Skiba points which can generate u-shaped optimal mitigation policies. Hence, we believe this represents an important area for further research.

[7] We have also made good use of lecture notes by Gerald North which can be found here http://www.met.tamu.edu/class/atmo631/atmo631.html.

[8] In contrast, a “zero-dimensional” model does not explicitly account for the spatial dimension while a one-dimensional model covers only latitudes. In Brock et al. (2013a) we develop a one-dimensional EBM coupled to an economic model. More complicated spatial structures would also include a vertical

formulation of a basic energy balance model, following (North et al., 1983) (Eq. (1)), and discuss how the solution can be pursued in terms of Fourier expansions in the eigenfunctions of the Laplacian operator. We will then proceed by deriving a more explicit solution to a simplified version of the model found in North et al. (1983), which features only a minimum of spatial complexities but with the added complication of human related forcing. Since, these models most likely are new to economist's we believe this to be an important first step in making these models easily accessible for economist's wishing to explore other aspects related to climate change and economic growth which we have not covered here. Also, adding further spatial complexity will require an extended numerical effort which lies beyond the scope of the present paper. The structure of the climate model we derive here lies close to several other models and developments found in e.g. North (1975a,b), North et al. (1981, 1983), North and Cahalan (1981), Kim and North (1991), Wu and North (2007).[9]

Let $\hat{r} = (\theta, \phi)$ denote a point on the surface of Earth where $\theta \in [-\pi/2, \pi/2]$ is latitude and $\phi \in [0, 2\pi]$ the longitude in radians.[10] Adding a radiative forcing component h_t which captures the human related activities influencing the climate such as e.g. the release of carbon dioxide into the atmosphere, following North et al. (1983) (Eq. (1)), we can write this model as

$$\mathrm{C}(\hat{r})\frac{T(\hat{r},t)}{dt} = QS(x)\alpha\left(x, T\left(\hat{r},t\right)\right) - I_{\hat{r},t} + h_t + \nabla\left[D(x)\nabla T\left(\hat{r},t\right)\right] \tag{1}$$

This equation is similar to the energy balance model we have derived in Brock et al. (forthcoming), Brock et al. (2013a) but with the extension that we are now dealing with a partial differential equation (PDE) that covers the entire surface of the Earth as opposed to only the latitude dimension we worked with earlier. The left hand side denotes the temperature differential between two points in time multiplied by the local heat capacity per unit area $\mathrm{C}(\hat{r})$, which takes on different values over e.g. oceans, land and perennial sea ice. The temperature differential at a specific point on the sphere is then determined by the differences in incoming and outgoing radiation together with the local heat diffusion across space. Defining $x = \sin\,\theta$ as the sine of the latitude, the incoming solar radiation hitting the surface of Earth, is then given by, $QS(x)\alpha(x, T(\hat{r},t))$, where Q is the solar constant divided by 4, $S(x)$ is the mean annual meridional distribution of solar radiation which by definition should be normalized so that its integral from -1 to 1 is unity; $\alpha(x, T(\hat{r},t))$ is the absorption coefficient or co-albedo function which is one minus the albedo of the earth–atmosphere system. $D(x)$ represents the diffusion term for all different forms of heat transport.[11] Here the $T(\hat{r},t)$ dependence captures variations in the iceline latitude, due to variations in local temperature.[12]

The outgoing radiation, is denoted by $I_{\hat{r},t}$ and determined by surface temperature $T(\hat{r},t)$ at a specific point on the sphere which can be estimated through the following empirical formula[13]

$$I_{\hat{r},t} = A + BT(\hat{r},t) \tag{2}$$

where the parameters A and B can be determined empirically by simple regression analysis. The heat transport (divergence in heat flux) term $\nabla[D(x)\nabla T(\hat{r},t)]$ is due the incoming absorbed radiant heat not being matched by the net outgoing radiation. ∇ is the spherical derivative operator in the variables, $x = \sin\,\theta$ and ϕ. In the case $D(x)$ is constant then $\nabla[D(x)\nabla T(\hat{r},t)] = D\nabla^2 T(\hat{r},t)]$ is the spherical Laplacian in (θ, ϕ). Eq. (1) thus states that the rate of change of in surface temperature is determined by the difference between the incoming absorbed radiant heat $QS(x)\alpha(x, x_{st})$ and the outgoing radiation $[A + BT(\hat{r},t)] - h_t$ where that the outgoing radiation is reduced by the accumulation of anthropogenic emissions of carbon dioxide h_t.

A crude way of capturing human related forcing from carbon dioxide emissions is $h_t = \xi\,\ln\left(M_t/M_{pre}\right)$ (see e.g. table 6.2 of the IPCC, 2001). Here M_{pre} denotes the pre-industrial atmospheric CO_2 concentration, M_t the stock of carbon dioxide in the atmosphere at time t. The coefficient ξ (W/m^2) is derived from detailed radiative transfer calculations (see e.g. Myhre et al., 1998).

(*footnote continued*)
dimensional describing how heat diffuses across different layers of the atmospheres and oceans. See e.g. Shell and Somerville (2005) for a model that has both a surface temperature and an atmospheric temperature but still retains much of the tractability of the simpler energy balance models.

[9] These models were developed in continuous time. Here we work out the corresponding model in discrete time. The discrete time framework offers a great deal of extra flexibility and analytical tractability compared to the continuous time framework which will become clear when we couple the climate model derived below to the economic models in the following sections.

[10] This is a slight abuse of language since longitude coordinates are typically defined in the range $[-\pi, \pi]$.

[11] A common parametrization as found in e.g. North et al. (1983) and Kim and North (1991) sets $D(x) = D_0(1 + D_2x^2 + D_4x^4)$ where the coefficients are adjustable parameters. Following the argument given by Lindzen and Farrell (1977) that the tropical Hadley cells are much more efficient at smoothing temperature anomalies than the midlatitude eddie, the coefficients are then adjusted to be three times as larger at the equator than at the poles.

[12] The iceline refers to the geographical position where polar ice caps start to form, We will sometimes throughout this paper abuse language and just refer to x as "latitude".

[13] It is important to note that the original Budyko (1969) formulation cited by North parameterizes A, B as functions of fraction cloud cover and other parameters of the climate system. North (1975b) points out that due to non-homogeneous cloudiness A and B should be functions of x. There is apparently a lot of uncertainty involving the impact of cloud dynamics (e.g. Trenberth et al., 2010 versus Lindzen and Choi, 2009). Hence robust control in which A, B are treated as uncertain may be called for but this is left for further research.

2.1. Approximating solutions for the basic energy balance equation

Having discussed the basic properties of our EBCM we now discuss how the solution to Eq. (1) can be approximated. Solutions to partial differential equations such as equation (1) are complex and require the use of numerical methods. This also implies a decision on what spatial discretization method to use. The two dominant methods used in solving large scale climate models are finite difference methods and spectral methods.[14] The approach we take on here follows the spectral solution method which is common in many global atmospheric circulation models since it offers several computational advantages compared to the finite difference method. These methods have also been applied successfully in the EBCM framework (see e.g. North and Cahalan, 1981; North et al., 1983; Kim and North, 1991; Wu and North, 2007). The basic idea behind spectral methods is to choose a finite set of basis functions and coefficients in order to approximate the exact solution as good as possible. These basis functions can be normalized to form an orthonormal basis of a Hilbert space, and are given by trigonometric functions, hence the name "spectral methods".

In order to find a solution to Eq. (1) we thus seek to expand $T(\hat{r},t)$, $\mathrm{C}_{\hat{r}}$ and $D(x)$ in terms of an appropriate basis function. As it turns out, spherical harmonics constitute such a basis function and are also eigenfunctions of the Laplace operator on the sphere. The spherical harmonics can be obtained as the product of a Fourier series in longitude and associated Legendre polynomials $P_n^m(x)$ in latitude. The expansion of $T(\hat{r},t)$ in spherical harmonics can be written as

$$T(\hat{r},t) = \sum_{n=0}^{\mathrm{N_T}} \sum_{m=-n}^{n} T_{n,m,t} \mathrm{Y}_{n,m}(\hat{r}), \tag{3}$$

and the expansion of $\mathrm{C}(\hat{r})$ as

$$\mathrm{C}(\hat{r}) = \sum_{n=0}^{\mathrm{N_C}} \sum_{m=-n}^{n} \mathrm{C}_{n,m} \mathrm{Y}_{n,m}(\hat{r}), \quad D(x) = \sum_{n=0}^{\mathrm{N_D}} D_n \mathrm{Y}_{n,0}(\hat{r}) \tag{4}$$

Inserting these approximations into Eq. (1) this transforms the partial differential equation (1) into a finite set of ordinary differential equations. Here, m refers to the longitudinal wave number and n the degree of the associated Legendre polynomial. The spherical harmonics are defined as

$$\mathrm{Y}_{n,m}(\hat{r}) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}} P_n^m(x) e^{-im\phi} \tag{5}$$

where $P_n^m(x)$ denotes the associated Legendre polynomial. Using the approximations (3) and (4) the solution to (1) can thus be approximated to an increasing degree of accuracy by letting N_T, N_C and N_D grow larger. For more details on using spectral methods for solving complex climate models excellent treatments can be found in textbooks such as Washington and Parkinson (2005) or Neelin (2011).

2.2. Approximating the solution of a simplified 2-D EBCM model

Since this paper is the first, to our knowledge that couples an economic growth model to a two-dimensional climate model we will proceed with a simplified version of (1) where the spatial complexities have been reduced to a minimum. By doing so, it is our hope that this makes the physical methods most easily accessible to economist's working on climate-economy type problems, thereby revealing the gist of spectral methods for solving these types of problems.[15] Despite the simplifications we still believe our results may be of practical importance for anyone wishing to carry on this work further with a more realistic model.

We will start by assuming that the co-albedo is quadratic in x and independent of the ice line $(\alpha(x,T(\hat{r},t)) = \alpha(x))$. Next, we ignore any elements of horizontal heat transport and simply assume it to be isotropic although we recognize that this is clearly not the case given the prevailing wind and ocean patterns. For the latitude dependence on we also ignore the latitude dependence in heat transport and assume a constant diffusion coefficient $(D(x) = D)$. We recognize that by doing so we fail to capture the fact that the tropical Hadley cells are much more efficient at smoothing temperature anomalies than the midlatitude eddies (Lindzen and Farrell, 1977). Finally, we assume a constant heat capacity $(\mathrm{C}(\hat{r}) = \mathrm{C})$. This is of course also a gross over simplification. The heat capacity measures the amount of heat needed to alter the temperature of a substance by a given amount. Geographically, the distribution of continents and ocean masses is important in determining the magnitude and spatial dependence of the heat capacity. Together, these simplifications allow for a transparent characterization of the dynamics of our climate system while still displaying the power of spectral solution methods for solving atmospheric climate models.[16]

[14] For a discussion on finite versus spectral methods see e.g. Neelin (2011) or Washington and Parkinson (2005).

[15] Based on preliminary attempts we have made with more spatially complex models such as North et al. (1983) (our Eq. (1)), we found that adding further complexity of this type still leaves the qualitative results found in this paper intact, the expressions will however become much more involved and for ease of exposition we thus avoid these complexities.

[16] The model is now close to the one derived in North and Cahalan (1981), with the exception that they model forcing as stochastic, which we leave for future work.

Before proceeding with the spatial expansion of Eq. (1) we start by discretizing the temporal dimension which is needed in order to couple the climate module to the economic growth model which we develop entirely in discrete time.

$$\mathrm{C}(T_{\hat{r},t+1}-T_{\hat{r},t}) = QS(x)\alpha(x)-[A+BT_{\hat{r},t}]+h_t+D\nabla^2 T_{\hat{r},t} \tag{6}$$

Here, the time step has been absorbed in the constant heat capacity coefficient C sometimes referred to by climate scientists as a phenomenological constant i.e. adjusted to fit observations.

Before expanding (6) in terms of spherical harmonics we note the orthogonality property of the normalized spherical harmonics (5)

$$\int_{-1}^{1}\int_{0}^{2\pi} Y_{n,m}Y^*_{n',m'}\,\mathrm{d}x\,\mathrm{d}\phi = \delta_{nn'}\delta_{mm'} \tag{7}$$

Here, the $Y^*_{n',m'}$ denotes the complex conjugate and $\delta_{nn'}\delta_{mm'}$ are the Kronecker delta's taking on a value of one for all $n=n'$ and $m=m'$ and zero otherwise.[17] This is an important property of the spherical harmonics which will simply the way in which human forcing impacts on the model.

In this case one sees right away that after substituting in the discrete time version of Eq. (3) i.e.

$$T_{\hat{r},t} = \sum_{n=0}^{\mathrm{N_T}}\sum_{m=-n}^{n} T_{n,m,t}\mathrm{Y}_{n,m}(\hat{r}), \tag{8}$$

into (6) and multiplying each side by the complex conjugate $\mathrm{Y}^*_{n',m'}(\hat{r})$, then by integrating over the whole globe and making use of Eq. (7) we arrive at the following expression[18]:

$$\mathrm{C}(T_{n,m,t+1}-T_{n,m,t}) = QH_{n,m}-\sqrt{4\pi}(A-h_t)\delta_{n,0}-(Dn(n+1)+B)T_{n,m,t} \tag{9}$$

for all $\{n,m\}$ where

$$H_{n,m} \equiv \int \mathrm{Y}^*_{n,m}(\hat{r})S(x)\alpha(x)\,\mathrm{d}\Phi \tag{10}$$

where we let $\int \equiv \int_{-1}^{1}\int_{0}^{2\pi}$ denote the integral over the whole globe and $\mathrm{d}\Phi = \mathrm{d}x\,\mathrm{d}\phi$ the differential volume element on the sphere. The key thing one notices from Eq. (9) is that due to the Kronecker delta $\delta_{n,0}$, being zero for all $n\neq 0$, the anthropogenic forcing h_t only impacts on the zero mode which was a result we also derived for the one-dimensional case we considered in Brock et al. (forthcoming), Brock et al. (2013a). Hence, using the above approximation methods we may represent any point on the sphere with an increasing degree of accuracy, the more terms included i.e. the larger is N in Eq. (8).

There is some arbitrariness to how one should truncate this series. In atmospheric science there have appeared two truncation conventions, the rhomboidal and the triangular. We will adopt a triangular truncation scheme in the numerical section of this paper, however for most numerical integration problems there is no compelling advantage (see e.g. Washington and Parkinson, 2005).

Next, the stock of the atmospheric carbon dioxide evolves according to

$$M_{t+1}-M_t = \eta\int q_{\hat{r},t}\,\mathrm{d}\Phi - mM_t + E_{land} \tag{11}$$

where M_t denotes the carbon dioxide stock and $\eta q_{x,t}$ are emissions generated at location $\hat{r}$, with emissions being proportional to the amount of fossil fuels used by location $\hat{r}$, at time t. Following Cai et al. (2012a), E_{land} denotes emissions from land being a natural source of emissions independent on the presence of humans.[19] The coefficient η reflects emission intensity of the fossil fuels. Additionally, a fraction m of the atmospheric carbon stock above pre-industrial carbon dioxide levels M_0 decays at each time step into the deep ocean.[20]

We assume that the total stock of fossil fuel available is fixed or,

$$\int q_{\hat{r},t}\mathrm{d}\Phi = q_t,\ \sum_{t=0}^{\infty} q_t \leq \mathcal{R}_0 \tag{12}$$

[17] Complex conjugates are pairs of complex numbers, having the same real part, but with imaginary parts of opposite signs but equal magnitude. For the spherical harmonics the complex conjugate can be found by noting that $Y_{n,-m} = (-1)^m Y^*_{n,m}$.

[18] To see this note that we have normalized so that $\mathrm{Y}_{0,0} = 1/\sqrt{4\pi}$.

[19] We recognize that humans can alter the emissivity of land and that is a major source of emissions. However, for the vastly simplified one-layer carbon cycle model we are considering here this assumption assures us that the model is well defined over all possible sets of $\{M(t)\}$. Note also that given the definition of M_{pre} (the pre-industrial carbon dioxide level), it is also natural to assume that if emissions $q_{\hat{r},t}$ go to zero then the resulting equilibrium level of atmospheric carbon dioxide will be $M_{pre} = E_{land}/m$, i.e. we get a return to pre-industrial levels.

[20] As mentioned in the footnote above, Eq. (11) seriously understates the complexity of the carbon cycle. For example, studies by Archer (2005) claims that "…75% of an excess atmospheric carbon concentration has a mean lifetime of 300 year and the remaining 25% stays there forever". This is cannot be captured by (11) and would require a more elaborate carbon cycle. This lies outside the scope of the current paper but constitutes an important area for future research.

where q_t is total fossil fuels used across all locations at time t, and $\mathcal{R}_0$ is the total available amount of fossil fuels on the planet. Thus in this model the use of fossil fuels generates emissions, emissions increase the stock of atmospheric carbon dioxide, which in turn increases the temperature by blocking the amount of outgoing radiation.

2.2.1. Global temperature means and variances

The approximating solution defines the climate module by (9), (8), (11), and (12). Although the climate module does not contain the PDE that incorporates temperature diffusion, spatial interactions are incorporated through the modes of the solution. The contribution of the modes greater than zero into the full solution can be regarded as the "importance of space" through heat transport, in the analysis of climate change. This can be seen by the following argument.

The size of diffusion coefficient D determines the speed of spatial diffusion. If $D=0$ then there are no spatial interactions, as $D\rightarrow\infty$ then we have an increase in mixing and spatial homogeneity and the heat transport across latitudes thus becomes less and less relevant for our problem. For a given diffusion $D<\infty$ the relative contribution of $T_{n,m,t}$ modes with $n>0$, to the solution of $\hat{T}_{\hat{r},t}$ can thus be regraded as a measure of whether the heat transport is important or not in the solution of the problem.

This suggests that the use of the global mean temperature alone in IAMs may introduce a bias. From the approximation of the temperature (8), we obtain the average global temperature as

$$\overline{T}_t=\frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}T_{\hat{r},t}\,d\Phi=T_{0,0,t}\mathrm{Y}_{0,0} \tag{13}$$

To see this start by inserting (8) into the double integral above. By the orthogonality property (7) all modes $n>0$ disappear. The result follows. This suggests that the zero dimensional IAMs can be regarded as a special case of the two-dimensional models when $D\rightarrow\infty$. Since scientific evidence indicates that D is small (less than one according to North et al., 1981) our result suggest that the dropping of the modes $n>0$, as is done in many IAMs introduces a kind of bias because cross-latitude externalities are ignored. In our paper we can correct for such a bias by increasing the amount of modes, which also provides a basis for a quantitative representation of this bias. The variance of the global mean temperature is

$$V(T)=\int_{-1}^{1}\int_{0}^{2\pi}[T_{\hat{r},t}-T_{0,0,t}]^2\mathrm{d}\Phi=\int_{-1}^{1}\int_{0}^{2\pi}\left(\sum_{n=0}^{\mathrm{N}}\sum_{m=-n}^{n}T_{n,m,t}\mathrm{Y}_{n,m}\right)^2\mathrm{d}\Phi=\sum_{n=0}^{\mathrm{N}}\sum_{m=-n}^{n}(T_{n,m,t})^2 \tag{14}$$

which also follows from the orthogonality property (7). In an IAM this variance will be zero since all modes greater than zero are dropped. Local temperature means over a limited area of the sphere Z is defined by

$$m(Z)=\frac{1}{Z}\int_Z\left[\sum_{n=0}^{\mathrm{N}}\sum_{m=-n}^{n}T_{n,m,t}\mathrm{Y}_{n,m}\right]\mathrm{d}\Phi \tag{15}$$

while the variance of temperature over the set of latitudes $Z=[a,b]$ is

$$V(Z)=\int_Z\left[\sum_{n=0}^{\mathrm{N}}\sum_{m=-n}^{n}T_{n,m,t}\mathrm{Y}_{n,m}-m(Z)\right]^2\mathrm{d}\Phi \tag{16}$$

It might be plausible to assume that utility in each area Z depends upon both the mean temperature and the variance of temperature in that area. For example we may expect increases in mean temperature and variance to have negative impacts on output in any area Z, if it is located in tropical latitudes. Whereas mean temperature increases in some areas Z (e.g. Siberia) may increase utility rather than decrease utility.[21]

3. An economic EBC model

In this section we characterize the solution to the planning problem. Based on the results of Section 2 this implies that a well defined planning problem should consider resource allocation across both time and space. We first describe the general features of the economy and then proceed with the problem of the planner. In the proceeding section we then characterize the market equilibrium.

[21] In a stochastic generalization of our model, we could introduce a stochastic process to represent "weather," i.e. very high frequency fluctuations relative to the time scales we are modeling here. Here the "local variance" of high frequency phenomena like "weather" may change with changes in lower frequency phenomena such as mean area Z temperature and area Z temperature variance. We leave this task to future research. Existing dynamic integrated models of climate and economy (e.g. Nordhaus's well known work (2007, 2010)) cannot deal with these kinds of spatial elements, such as impacts of changes in temperature variance, generated by climate dynamics over an area Z.

3.1. General features of the economy

The economy is assumed to be inhabited by a representative household at location $\hat{r}$ having preferences defined by the following utility function.

$$U(C_{\hat{r},t}/L_{\hat{r},t}) = \frac{(C_{\hat{r},t}/L_{\hat{r},t})^{1-\nu}-1}{(1-\nu)} \tag{17}$$

where $C_{\hat{r},t}$ denotes aggregate consumption and $L_{\hat{r},t}$ the size of the representative household (equal to population) at time t and for point $\hat{r}$ on the sphere. U is thus a standard concave utility function having a constant intertemporal elasticity of substitution $1/\nu$. Labor is supplied inelastically and is equal to population, which grows at a constant factor $n \geq 1$ across all locations so that $L_{\hat{r},t} = L_{\hat{r},0}n^{t-1}$. Production takes place separately at each location

$$Y_{\hat{r},t} = \mathcal{A}_{\hat{r},t}\Omega(T_{\hat{r},t})F(K_{\hat{r},t}, L_{\hat{r},t}, q_{\hat{r},t}) \tag{18}$$

where $K_{\hat{r},t}, L_{\hat{r},t}, q_{\hat{r},t}$ denote capital, labor and fossil fuels respectively used at point $\hat{r}$, and time t, $a \geq 1$ is the TFP growth so that $\mathcal{A}_{\hat{r},t} = \mathcal{A}_{\hat{r},0}a^{t-1}$, and $\Omega(T_{\hat{r},t})$ are the damages to output due to climate change occurring at point $\hat{r}$ and time t as a function of the local temperature at the that location, with $(\partial\Omega(T_{\hat{r},t})/\partial T_{\hat{r},t}) < 0$. We will throughout assume that production function $F(\cdot)$ exhibits constant returns to scale. The total amount of fossil fuels available is finite and given by Eq. (12) of the previous section. The spatial aspect of our model also induces the possibility to consider the possibility of intertemporal trade. We denote the net stock of foreign bond or asset holdings at location $\hat{r}$ by $B_{\hat{r},t}$ which generate a return R_t. Negative values thus imply that point $\hat{r}$ has an outstanding debt to other locations. The budget constraint of each country is thus given by

$$C_{\hat{r},t} + K_{\hat{r},t+1} + B_{\hat{r},t+1} \leq Y_{\hat{r},t} + (1-\delta)K_{\hat{r},t} + R_t B_{\hat{r},t} \tag{19}$$

from which we can write the global budget constraint by integrating over x

$$C_t + K_{t+1} \leq Y_t + (1-\delta)K_t \tag{20}$$

where

$$0 = \int B_{\hat{r},t}\, d\Phi, \quad C_t = \int C_{\hat{r},t}\, d\Phi, \quad K_t = \int K_{\hat{r},t}\, d\Phi, \quad Y_t = \int Y_{\hat{r},t}\, d\Phi \tag{21}$$

3.2. Global welfare maximization

Given the description of the economy and climate dynamics provided so far, it is clear that the welfare maximization problem of a planner, should involve both spatial and intertemporal decision making. We formalize this by defining a global social welfare function where each individual country is pre-assigned a specific welfare weight, so that by varying these weights we can trace out different distributional outcomes for the global economy. The welfare function of the planner can be written as

$$\sum_{t=0}^{\infty} \beta^t \int \upsilon(\hat{r}) L_{\hat{r},t} U(C_{\hat{r},t}/L_{\hat{r},t})\, d\Phi \tag{22}$$

where β represents the discount factor and $\upsilon(\hat{r}) \geq 0$ denotes the location specific welfare weights.[22] An early result noted by Chichilnisky and Heal (1994) shows that in a multi-country optimal planning problem featuring both a private and a public good, whether the solution to the problem will equalize marginal abatement costs across countries will largely depend upon whether international transfers are available or not.[23] In the absence of such transfers a spatially uniform carbon tax is in general not optimal.[24] Within the two-dimensional energy balance model we consider here, the heat diffusion across locations creates a spatially differentiated damage function which makes the results of Chichilnisky and Heal (1994) increasingly relevant. In particular, since international transfers may many times prove to be hard to implement in practice.[25] We consider two specific cases.

[22] How, welfare weights are set has large implications for the distribution of income and wealth. For example, within our framework, if welfare weight are set so that $\upsilon(\hat{r}) = 1/(4\pi L_{\hat{r},0})^{\nu}\ \forall \hat{r}$, this would imply equal weighting of all locations regardless of population size so that a location with a population the size of China would be equally weighted as a location with only a very small population, such as Malta. Such a weighting scheme would by many people seem unjust since it would imply that huge transfers be made from China to Malta. An alternative weighting scheme typically referred to as Negishi weighting (Negishi, 1960) assigns welfare weights so as to avoid that large transfers takes place across regions. Many different weighting schemes are of course possible and as pointed out by Stanton (2010) these assumptions will have implications also for optimal climate policy. An assessment of such implications lies beyond the scope of the current paper. For a recent analysis of inequality within the context of an IAM see Dennig (2013).

[23] Similar results have also been obtained by Chichilnisky et al. (2000), Shiell (2003b), Sandmo (2006), Anthoff (2011), Keen and Kotsogiannis (2011).

[24] Transfers of this kind are significant and observed in the real world not only in terms of foreign aid and development assistance but also in terms of payments for the supply of public goods examples include the Joint implementation and Clean development mechanisms incorporated into the Kyoto protocol. However, it is also true that countries to a great extent are limited by their own resources and incomes.

[25] Shiell (2003a) points to e.g. to corruption as one possibility that can make large international transfers unfeasible. In this case, the opportunity to help the poor through untied transfers may be seriously compromised by misappropriation or theft by officials who are charged with administering the transfers in recipient countries.

In the first case, which we will refer to as the *open economy* problem, the economy is completely open with free flows of capital, fossil fuel and consumption goods across locations. This implies that returns to capitals and bonds will be equal across locations.[26] The planner has the power to administer lump-sum transfers of goods across locations which implies that all Pareto optimal allocations corresponding to a particular set of welfare weights can be traced out. Due to his ability to freely transfer goods across locations the investment decisions of this planner is thus not limited by the capital budget constraint (19) involving each individual location but rather the sum, or as we have chosen to model it here, the integral over all location specific capital budget constraints.

Turning to the second case. We refer to this case as the *closed economy* problem since each location is limited by its own budget constraint. As will be shown, in this case, depending on whether the planner has access to international transfers or not, optimal tax rates may differ across locations. The particular assumptions connected to this scenario are restrictive and perhaps not so realistic but they help to bring out the forces that can generate spatially differentiated tax rates. In particular we show that if wealth transfers are restricted and welfare weights are assumed to be equal across locations this implies that optimal tax rates will be spatially differentiated.

We start by solving the *open economy* problem involving free international transfers. We then characterize the competitive equilibrium and derive the optimal tax rates which implement the planning solution. In the following section we will explore then explore *closed economy*.

3.3. Optimal planning problem with open markets and free international transfers

Given the economy and climate dynamics described above we start by considering the welfare maximization problem of an *open economy* where the planner has free access to international transfers across locations. In this case the planner maximizes the welfare function defined in Eq. (22) subject to (8), (9), (11), (12), (20), and (21). As can be seen from inspection of this maximization problem, it can be broken down into a static and dynamic maximization problem which can be considered separately. We start by characterizing the static problem.

3.3.1. Static problem

The spatial allocation decision facing the planner involves the allocation of consumption, capital and fossil fuels to each location so as to maximize global output and social welfare in each time period subject to the global capital budget constraint (20). As will be seen the solution to this problem implies that the marginal products of $K_{\hat{r},t}$ and $q_{\hat{r},t}$ must be equal across locations and that the weighted marginal utility of $C_{\hat{r},t}$ at each location must also be equal.

First, in order to determine how capital and fossil fuels are allocated across locations given the global intertemporal investment and fossil fuel use decisions governed by the capital budget constraint, we proceed by solving the following problem; assuming a Cobb–Douglas production function[27]:

$$\mathbb{Y}(K_t, q_t, \{T_{\hat{r},t}\}_{\hat{r}}, t) \equiv \max_{\{K_{\hat{r},t}, q_{\hat{r},t}\}_{\hat{r}}} \left\{ \int (an^{\alpha_L})^{t-1} \mathcal{A}_{\hat{r},0} L_{\hat{r},0}^{\alpha_L} \Omega(T_{\hat{r},t}) K_{\hat{r},t}^{\alpha_K} q_{\hat{r},t}^{\alpha_q} \, \mathrm{d}\Phi \right\}$$

$$\text{s.t.} \quad K_t \geq \int K_{\hat{r},t}/K_t \, \mathrm{d}\Phi, \; q_t \geq \int q_{\hat{r},t}/q_t \, \mathrm{d}\Phi \tag{23}$$

We define $\mathbb{Y}(K_t, q_t, \{T_{\hat{r},t}\}_{\hat{r}}, t)$ as the "*potential world GDP at datet*". This concept represents the maximum output that the whole world can produce given total world capital K_t available and total world fossil fuel q_t used, for a given distribution of temperature $T_{\hat{r},t}$ across the globe, with labor growing with the constant factor n, and treated as immobile. Thus $\mathbb{Y}$ can be regarded as a natural base line under ideal world conditions where there's no barriers to capital and fossil fuel flows to their most productive uses across locations.[28] Next, define the shares of K and q at date t by $S^K_{\hat{r},t} \equiv K_{\hat{r},t}/K_t$ and $S^q_{\hat{r},t} \equiv q_{\hat{r},t}/q_t$. It is now easy to check that the optimal shares solving (23) will be given by[29]

$$S^K_{\hat{r},t} = S^q_{\hat{r},t} = (\mathcal{A}_{\hat{r},0} L_{\hat{r},0}^{\alpha_L} \Omega(T_{\hat{r},t}))^{1/\alpha_L} \Big/ \int (\mathcal{A}_{\hat{r},0} L_{\hat{r},0}^{\alpha_L} \Omega(T_{\hat{r},t}))^{1/\alpha_L} \, \mathrm{d}\Phi \tag{24}$$

Substituting the optimal shares back into the objective function we obtain the following expression for potential world GDP:

$$\mathbb{Y}(K_t, q_t, \{T_{\hat{r},t}\}_{\hat{r}}, t) = (an^{\alpha_L})^{t-1} K_t^{\alpha_K} q_t^{\alpha_q} \left(\int (\mathcal{A}_{\hat{r},0} L_{\hat{r},0}^{\alpha_L} \Omega(T_{\hat{r},t}))^{1/\alpha_L} \, \mathrm{d}\Phi \right)^{\alpha_L} \tag{25}$$

[26] We neglect migration and assume labor is completely immobile across locations. Labor immobility at a global scale could be regarded as a reasonable approximation given restrictions on labor mobility relative to capital and fossil fuel mobility.

[27] Hassler et al. (2012) argue that, on shorter time horizons, Cobb–Douglas production does not represent a good way of modeling energy demand since it does not capture the joint shorter- to medium-run movements of input prices and input shares. However, on the longer time horizons we consider here it is more reasonable since input shares do not appear to trend because input shares have not appeared to have trended in the past. But there appears to be some falling in labor's share recently. But it is beyond the scope of this paper to discuss this issue further.

[28] This notion can be regarded as similar to the notions of "potential GDP" "potential output" etc. used by macro economists.

[29] See Appendix A.1 for a complete derivation.

As it can be seen from (25) the Cobb–Douglas specification allows the "separation" of the climate damage effects on production across locations. Define

$$J(\{T_{\hat{r},t}\}_{\hat{r}}) \equiv \left(\int (\mathcal{A}_{\hat{r},0}L_{\hat{r},0}^{\alpha_L}\Omega(T_{\hat{r},t}))^{1/\alpha_L}\,\mathrm{d}\Phi\right)^{\alpha_L} \tag{26}$$

This expression depends on the thermal diffusion coefficient D which multiplies a production function that is independent of $\hat{r}$. Thus population growth and technical change affect the "macrogrowth component" $(an^{\alpha_L})^{t-1}K_t^{\alpha_K}q_t^{\alpha_q}$, while changes in the size of D have a direct effect on the "climate component". The combination of the macrogrowth and the climate component determine the potential world input.

Second, the allocation consumption across locations is solved in an equivalent manner by

$$\mathbb{U}(C_t) \equiv \max_{\{C_{\hat{r},t}\}_{\hat{r}}} \int \upsilon(\hat{r})L_{\hat{r},t}U(C_{\hat{r},t}/L_{\hat{r},t})\,\mathrm{d}\Phi,$$
$$\text{s.t.}\quad C_t \geq \int C_{\hat{r},t}\,\mathrm{d}\Phi \tag{27}$$

Hence, $C_{\hat{r},t}$ is restricted by the total amount of available consumption C_t which in turn is restricted by the global capital budget constraint. The first order conditions with respect to $C_{\hat{r},t}$ thus implies

$$\nu(\hat{r})U'(C_{\hat{r},t}/L_{\hat{r},t}) = \nu(\hat{r}')U'(C_{\hat{r}',t}/L_{\hat{r}',t}) \tag{28}$$

Define the location specific optimal consumption shares $S_{\hat{r},t}^C \equiv C_{\hat{r},t}/C_t$. By (28) and (21) optimal shares are thus given by[30]

$$S_{\hat{r},t}^C \equiv \nu(\hat{r})^{1/\nu}L_{\hat{r},t}\Big/\int \nu(\hat{r}')^{1/\nu}L_{\hat{r}',t}\,\mathrm{d}\Phi' \tag{29}$$

where $-\nu$ denotes the elasticity of marginal utility of consumption which follows from (17).

From this expression we see how the right hand side of Eq. (29) determines the optimal share of aggregate consumption that is allocated to location $\hat{r}$. The size of the denominator depends also on how we normalize welfare weights which may be done in various ways without impacting on distribution. A common way of doing this in the absence of population is to set $\int \upsilon(\hat{r})\,\mathrm{d}\Phi = 1$. However, in our case, since we have population growth, we have chosen to normalize welfare weights so that $\int \upsilon(\hat{r})^{1/\nu}L_{\hat{r},0}\,\mathrm{d}\Phi = L_0$, where L_0 denotes the world population at $t=0$. Hence, if we define $S_{\hat{r},0}^L \equiv L_{\hat{r},0}/L_0$ then we can further write $\int \upsilon(\hat{r})^{1/\nu}S_{\hat{r},0}^L\mathrm{d}\Phi = 1$. Finally, as pointed out in the previous section we assume that population growth rates n, are equal across locations. This way of normalizing welfare weights thus implies that the relationship $\int \upsilon(\hat{r})^{1/\nu}L_{\hat{r},t}\mathrm{d}\Phi = L_t$ also holds which further implies that the denominator of (29) will equal L_t. As will be seen in the following section, this particular normalization scheme although arbitrary, eases up on notation substantially. The following relationship between per capita consumption in location $\hat{r}$ and aggregate per capita consumption thus holds for any arbitrary set of predefined welfare weights $\upsilon(\hat{r}) > 0$:

$$C_{\hat{r},t}/L_{\hat{r},t} = \nu(\hat{r})^{1/\nu}(C_t/L_t) \tag{30}$$

or alternatively we can express this in terms of marginal utilities

$$U'(C_{\hat{r},t}/L_{\hat{r},t}) = \nu(\hat{r})^{-1}U'(C_t/L_t) \tag{31}$$

3.3.2. *The dynamic problem*

Given the static solutions derived above we can now write the dynamic welfare maximizing problem of the planner in for an arbitrary set of welfare weights and in terms of potential world GDP as follows:

$$\max \sum_{t=0}^{\infty} \beta^t \mathbb{U}(C_t),$$
$$\text{s.t.}\quad C_t + K_{t+1} \leq \mathbb{Y}_t + (1-\delta)K_t \tag{32}$$

and (8), (9), (11), and (12). Here, we have replaced the social welfare and production functions by maximized values following the static efficiency condition of the previous section.

The first order necessary conditions w.r.t. C_t, K_{t+1} gives us a standard Euler equation

$$\mathbb{U}'(C_t) = \beta\mathbb{U}'(C_{t+1})\left(\frac{\partial \mathbb{Y}_{t+1}}{\partial K_{t+1}} + (1-\delta)\right) \tag{33}$$

Since the problem is concave in K this is sufficient for optimum over K.[31] Turn now to optimization w.r.t. q. At each date t a small increase in q_t gets a marginal gain in output at t but sets of a sequence of marginal changes in damages for periods

[30] See Appendix A.1 for a complete derivation of this problem.

[31] The solution to the static problem, derived in section Appendix A.1, implies that $\mathbb{U}'(C_t) = (C_t/L_t)^{-\nu}$. Had we not invoked the static efficiency condition w.r.t. consumption, Eq. (33) would instead have read $\nu(\hat{r})U'(C_{\hat{r},t}/L_{\hat{r},t}) = \beta\nu(\hat{r})U'(C_{\hat{r},t+1}/L_{\hat{r},t+1})((\partial\mathbb{Y}_{t+1}/\partial K_{t+1}) + (1-\delta))\ \forall \hat{r}$. The way we have written it here shortens notation and simplifies comparisons to other models, derived for single region settings such as e.g. Golosov et al. (2011).

$t+1, t+2, \ldots$. The first order necessary conditions for q_t at date t is given by[32]

$$\beta^t \mathbb{U}'(C_t)\frac{\partial \mathbb{Y}}{\partial q_t} = -\sum_{k=2}^{\infty} \beta^{t+k}\mathbb{U}'(C_{t+k})\frac{\partial \mathbb{Y}_{t+k}}{\partial T_{\hat{r},t+k}}\left(\sum_{s=1}^{k-1}\frac{\partial T_{\hat{r},t+k}}{\partial M_{t+s}}\frac{\partial M_{t+s}}{\partial q_t}\right)+\mu_0 \tag{34}$$

This expression is fundamental to our paper as it characterizes how the spatial aspects affect optimal fossil fuel use. It is also straightforward to interpret. The first order condition simply states that the marginal social utility of using an extra unit of fossil fuel (left hand side) must equal the marginal social cost (right hand side). The marginal social cost consists of both the capitalized sum of marginal damages to output at $t+1, t+2, \ldots$ plus the marginal cost of not having that unit of fossil fuel available in the future (when $\mathcal{R}_0$ is ultimately all used up).[33]

Alternatively, the capitalized sum of marginal damages could have been expressed as in Golosov et al. (2011) in terms of a shadow price (λ^s)

$$\lambda_t^m \equiv -\sum_{k=2}^{\infty} \beta^{t+k}\mathbb{U}'(C_{t+k})\frac{\partial \mathbb{Y}_{t+k}}{\partial T_{\hat{r},t+k}}\left(\sum_{s=1}^{k-1}\frac{\partial T_{\hat{r},t+k}}{\partial M_{t+s}}\frac{\partial M_{t+s}}{\partial q_t}\right) \tag{35}$$

or alternatively, in discounted per capita consumption units

$$\Lambda_t^m \equiv -\frac{1}{\beta^t \mathbb{U}'(C_t)}\sum_{k=2}^{\infty} \beta^{t+k}\mathbb{U}'(C_t)\frac{\partial \mathbb{Y}_{t+k}}{\partial T_{\hat{r},t+k}}\left(\sum_{s=1}^{k-1}\frac{\partial T_{\hat{r},t+k}}{\partial M_{t+s}}\frac{\partial M_{t+s}}{\partial q_t}\right) \tag{36}$$

Eq. (36) is thus our analog of equation 11 in Golosov et al. (2011). We can now from Eq. (34) write down a spatial version of the climate externality adjusted Hotelling formula

$$\frac{\mathbb{U}'(C_t)}{\beta \mathbb{U}'(C_{t+1})} = \frac{\frac{\partial \mathbb{Y}_{t+1}}{\partial q_{t+1}} - \Lambda_{t+1}^m}{\frac{\partial \mathbb{Y}}{\partial q_t} - \Lambda_t^m} \tag{37}$$

which by the Euler equation (33) above we know equals the return on investments or equivalently the real interest rate. Hence, once again we see how the return to postponing extraction equals the return on capital investments. The zero mode dynamics can be written as

$$\mathbb{C}(T_{0,0,t+1}-T_{0,0,t}) = QH_{0,0} - \sqrt{4\pi}(A+h_t) - BT_{0,0,t} \tag{38}$$

with the approximating temperature at each location determined by $T_{\hat{r},t} = T_{0,0,t}Y_0^0 + \sum_{(n,m)\neq(0,0)} T_{n,m,t}Y_n^m(\hat{r})$ where all modes higher than $(0,0)$ are thus independent of carbon dioxide M. This implies that $(\partial \mathbb{Y}_{t+k}/\partial T_{\hat{r},t+k})(\partial T_{\hat{r},t+k}/\partial M_{t+k-1}) = (\partial \mathbb{Y}_{t+k}/\partial T_{0,t+k})(\partial T_{0,0,t+k}/\partial M_{t+k-1})$ which greatly simplifies the analysis while still maintaining implicitly the temperature at each location.

To proceed further with analytical results we make some more assumptions in line with (Golosov et al., 2011), i.e. we assume log utility, Cobb–Douglas production, exponential damages and full depreciation of capital, which implies that the Euler equation (33) can be solved for the optimal investment and consumption rate. It follows immediately that the following rules solves the optimal saving decisions.

$$K_{t+1} = \alpha_K \beta \mathbb{Y}, \quad C_t = (1-\alpha_K\beta)\mathbb{Y} \tag{39}$$

Next, given an exponential damage function $\Omega(T_{\hat{r},t}) = e^{-\gamma T_{\hat{r},t}}$ damages can be written in the following multiplicative form $\Omega(T_{\hat{r},t}) = \Omega(T_{0,0,t}Y_0^0 + \Gamma_{\hat{r},t}) = \Omega(T_{0,0,t}Y_0^0)\Omega(\Gamma_{\hat{r},t})$ where $\Gamma_{\hat{r},t} \equiv \sum_{(n,m)\neq(0,0)} T_{n,m,t}Y_n^m(\hat{r})$. Under these assumptions we may now write (26) as

$$\begin{aligned} J(\{T_{\hat{r},t}\}_{\hat{r}}) &= \left(\int \Omega(T_{0,0,t}Y_0^0 + \Gamma_{\hat{r},t})^{1/a_L} A_{\hat{r},0}^{1/a_L} L_{\hat{r},0}\, \mathrm{d}\Phi\right)^{a_L} \\ &= \frac{\Omega(T_{0,0,t}Y_0^0)}{Y_0^0}\left(\int \Omega(\Gamma_{\hat{r},t})^{1/a_L} A_{\hat{r},0}^{1/a_L} L_{\hat{r},0}\, \mathrm{d}\Phi\right)^{a_L} \end{aligned} \tag{40}$$

Hence, we have

$$\frac{\partial \mathbb{Y}_{t+k}}{\partial T_{0,0,t+k}} = \frac{\partial \mathbb{Y}_{t+k}}{\partial T_{0,0,t+k}}\frac{\mathbb{Y}_{t+k}}{\mathbb{Y}_{t+k}} = \frac{\Omega'(T_{0,0,t})}{\Omega(T_{0,0,t})}\mathbb{Y}_{t+k} = -\gamma \mathbb{Y}_{t+k} \tag{41}$$

substituting this expression into (36) and making use of (39) we have

$$\Lambda_t^m = \gamma \mathbb{Y}_t \sum_{k=2}^{\infty} (\beta n)^{t+k}\left(\sum_{s=1}^{k-1}\frac{\partial T_{0,0,t+k}}{\partial M_{t+s}}\frac{\partial M_{t+s}}{\partial q_t}\right) \tag{42}$$

[32] Here, we have also made replaced the weighted marginal utilities at locations $\hat{r}$ by their maximized values following the static efficiency condition given in Eq. (31).

[33] When there are damages to consumption, there will be an extra term on the right hand side representing the capitalized sum of marginal damages to future social utility of climate during the following dates. This term can be potentially important for optimal mitigation decisions as shown in Sterner and Persson (2008). For simplicity we abstract from this here.

Thus we see that marginal damages as a fraction of GDP, $\Lambda_t^s/\mathbb{Y}_t$, is given by a very simple formula because $\partial M_{t+s}/\partial q_t$ is a constant, and if the dynamics of $T_{0,t}$ are forced by a constant times M_t then $\partial T_{0,0,t+k}/\partial M_{t+s}$ is a constant and if the dynamics are forced by log function as defined in Section 2 then $\partial T_{0,0,t+k}/\partial M_{t+s}$ is a constant divided by M_{pre}. Finally the Hotelling equation becomes completely independent of capital investment and takes the simple form.

$$\frac{1}{\beta n}=\frac{\alpha_q\dfrac{1}{q_{t+1}}-\gamma\sum_{k=2}^{\infty}(\beta n)^k\left(\sum_{s=1}^{k-1}\frac{\partial T_{0,0,t+1+k}}{\partial M_{t+1+s}}\dfrac{\partial M_{t+1+s}}{\partial q_{t+1}}\right)}{\alpha_q\dfrac{1}{q_t}-\gamma\sum_{k=2}^{\infty}(\beta n)^k\left(\sum_{s=1}^{k-1}\frac{\partial T_{0,0,t+k}}{\partial M_{t+s}}\dfrac{\partial M_{t+s}}{\partial q_t}\right)} \tag{43}$$

Hence, we see that given the stylized assumptions above we arrive at a simple expression for optimal fossil fuel use, independent of decisions on saving.

3.4. The market equilibrium and fossil fuel taxes

In this section we characterize the decentralized market equilibrium which given an appropriate set of transfers can implement the planning solution described above. As will be shown this is possible given that taxes rates are set equal to the emission externality. We start by characterizing the market equilibrium with taxes and then discuss optimal taxation within this setup. In the following, we implicitly assume the existence of a government that not only imposes externality taxes on location $\hat{r}-$firms but also transfers resources. The government must thus satisfy its own budget constraint with taxes collected on the income side and lump sum redistributions on the outgoing side. Consumers at location $\hat{r}$ are assumed to have full ownership of the goods producers at their location to whom they rent out capital and labor services. In excess to this, consumers at location $\hat{r}$ also own a fixed share $s_{\hat{r}}$ of a representative competitive fossil fuel firm extracting from a single finite stock of fossil fuel reserves. Consumers thus also collect a share $s_{\hat{r}}$ of the profits earned from fossil fuel sales by the representative firm.

3.4.1. Consumers

Consumers at location $\hat{r}$ can borrow and lend on world bond markets at the gross interest rate R_t to solve the following problem:

$$\max \sum_{t=0}^{\infty}\beta^t L_{\hat{r},t}U(C_{\hat{r},t}/L_{\hat{r},t}) \tag{44}$$

subject to

$$\sum_{t=0}^{\infty}p_t^k(C_{\hat{r},t}+K_{\hat{r},t+1}+B_{\hat{r},t+1})=\sum_{t=0}^{\infty}p_t^k(R_{\hat{r},t}K_{\hat{r},t}+R_tB_{\hat{r},t}+I_{\hat{r},t}) \tag{45}$$

$$B_{\hat{r},0}=B_{\hat{r}0},K_{\hat{r},0}=K_{\hat{r}0} \tag{46}$$

$$I_{\hat{r},t}\equiv w_{\hat{r},t}L_{\hat{r},t}+s_{\hat{r}}p_tq_t+\tau_{\hat{r},t}q_{\hat{r},t} \tag{47}$$

where $B_{\hat{r},t}$ denotes the amount of foreign bonds held at location $\hat{r}$ and time t.[34] The competitive price of consumption goods at time t is denoted by p_t^k. Labor $L_{\hat{r},t}$ is perfectly inelastically supplied at the wage rate $w_{\hat{r},t}$. Profits from fossil fuel firms (Hotelling rents) and taxes on fossil fuel use in production expressed in consumption prices are redistributed lump sum to location $\hat{r}$ consumers in the fraction $s_{\hat{r}}p_tq_t$ and $\tau_{\hat{r},t}q_{\hat{r},t}$ respectively. $R_{\hat{r},t}$ and R_t denote the rental rate of capital and the return on government bonds respectively defined in terms of consumption prices.[35] Further, the transversality conditions of capital and bonds are given by

$$\lim_{t\to\infty}\beta^tp_t^kB_{\hat{r},t+1}=0,\quad \lim_{t\to\infty}\beta^tp_t^kK_{\hat{r},t+1}=0 \tag{48}$$

From the first order necessary condition for consumption, capital and bonds we have

$$\beta^tU'(C_{\hat{r},t}/L_{\hat{r},t})=\Lambda_{\hat{r}}^kp_t^k \tag{49}$$

$$R_{\hat{r},t+1}=\frac{p_t^k}{p_{t+1}^k}=R_{t+1} \tag{50}$$

[34] $B_{\hat{r}0}$ and $K_{\hat{r}0}$ denote the initial amounts of bonds and capital holdings by location $\hat{r}$ at the outset. We further assume that initial bond holdings are zero for all locations $\hat{r}$.

[35] In baseline analysis using Arrow Debreu private ownership economies, it is standard to assume perfect markets (borrowing and lending with no frictions, defaults, etc.) with profits and taxes redistributed lump sum to consumers. Also note that the above quantities, need to be multiplied by p_t^k in order to express them in term of consumption prices.

where $\Lambda_{\hat{r}}^k$ denotes the Lagrange multiplier on capital for location $\hat{r}$ and the second equality in (50) follows from world interest rate on bonds. Iterating forward on Eq. (50) implies that consumption prices can also be expressed as $p_t^k = \prod_{i=1}^t R_i^{-1} p_0^k$ where we may normalize $p_0^k = 1$ so that the prices will be expressed relative to $t=0$ consumption goods.

Alternatively, we have could expressed (49) and (50) in recursive form as

$$\frac{U'(C_{\hat{r},t}/L_{\hat{r},t})}{\beta U'(C_{\hat{r},t+1}/L_{\hat{r},t+1})} = R_{t+1} \tag{51}$$

3.4.2. Consumption goods producing firms

Consumption goods producing firms at location $\hat{r}$ solve the following problem:

$$\max_{K_{\hat{r},t},L_{\hat{r},t},q_{\hat{r},t}} \{p_t^k[\mathcal{A}_{\hat{r},t}\Omega(T_{x,t})F(K_{\hat{r},t},L_{x,t},q_{\hat{r},t}) - (R_{\hat{r},t}-(1-\delta))K_{\hat{r},t} - w_{\hat{r},t}L_{\hat{r},t} - (p_t+\tau_{\hat{r},t})q_{\hat{r},t}]\} \tag{52}$$

where p_t denotes the world price fossil fuels plus the unit tax $\tau_{\hat{r},t}$ levied at x at date t, $w_{\hat{r},t}$ is the wage at location $\hat{r}$ and time t.[36] $F(K,L,q)$ is constant returns to scale, hence profits will be zero at each $\hat{r}$ for firms that produce consumption goods. The first order conditions for (52) are

$$\mathcal{A}_{\hat{r},t}\Omega(T_{\hat{r},t})F_K'(K_{\hat{r},t},L_{\hat{r},t},q_{\hat{r},t}) = R_{\hat{r},t}-(1-\delta) \tag{53}$$

$$\mathcal{A}_{\hat{r},t}\Omega(T_{\hat{r},t})F_L{}'(K_{\hat{r},t},L_{\hat{r},t},q_{\hat{r},t}) = w_{\hat{r},t} \tag{54}$$

$$\mathcal{A}_{\hat{r},t}\Omega(T_{\hat{r},t})F_q'(K_{\hat{r},t},L_{\hat{r},t},q_{\hat{r},t}) = p_t+\tau_{\hat{r},t} \tag{55}$$

Thus in any decentralized problem location $\hat{r}$ firms will choose demands $K_{\hat{r},t}$, $L_{\hat{r},t}$ and $q_{\hat{r},t}$ according to (53)–(55). Note that the marginal product of $q_{\hat{r},t}$ is equated across $\hat{r}$'s for every date t only if taxes on fossil fuels are equal across locations so that $\tau_{\hat{r},t} = \tau_t$.

3.4.3. Fossil fuel firm

A representative competitive fossil fuel firm maximizes the discounted sum of profits under zero extraction costs and solves the following problem

$$\max_{q_{\hat{r},t}} \sum_{t=0}^{\infty} p_t^k p_t q_{\hat{r},t}, \quad \text{s.t.} \sum_{t=0}^{\infty} \int q_{\hat{r},t}\, d\Phi \le \mathcal{R}_0 \tag{56}$$

with first order conditions

$$p_t^k k p_t = \mu_0 \tag{57}$$

where μ_0 denotes the Lagrange multiplier on the resource constraint. Alternatively, in recursive form we have

$$p_{t+1} = R_{t+1}p_t \tag{58}$$

3.5. Optimal carbon taxes in the open economy

Given the competitive market equilibrium described above we now proceed by showing that given an appropriately defined tax the competitive equilibrium described above can implement the solution to the planning problem. We start by defining an optimal tax rate $\tau_{\hat{r},t}^*$. From the firms optimality conditions w.r.t fossil fuel use (55) we see that it is only by setting $\tau_{\hat{r},t} = \tau_t^*$, that these conditions become equal across all locations. Hence, given a uniform tax τ_t^* and interest rate R_t, it thus follows that the shares of capital and fossil fuels in the competitive equilibrium will also be allocated according to $K_{\hat{r},t} = S_{\hat{r},t}^K K_t$ and $q_{\hat{r},t} = S_{\hat{r},t}^q q_t$ as in the planning problem where $S_{\hat{r},t}^K$ and $S_{\hat{r},t}^q$ are defined in expression (24) of the planning problem. Comparing the competitive first order condition with the planning solution it also follows that $\partial Y_{\hat{r},t}/\partial q_{\hat{r},t} = \partial \mathbb{Y}_t/\partial q_t$ and $\partial Y_{\hat{r},t}/\partial K_{\hat{r},t} = \partial \mathbb{Y}_t/\partial K_t$. This further implies due to the Euler (33) of the planning problem and condition (50), (51), and (53) of the competitive equilibrium that $U'(C_{\hat{r},t}/L_{\hat{r},t})/\beta U'(C_{\hat{r},t+1}/L_{\hat{r},t+1}) = \mathbb{U}'(C_t)/\beta \mathbb{U}'(C_{t+1})$ must also hold.

Turn now to the optimality conditions w.r.t. fossil fuels. Solving (49) for p_t^k and (55) for p_t and substituting these expressions into Eq. (57) we have

$$\frac{1}{\Lambda_{\hat{r}}^k}\beta^t U'(C_{\hat{r},t}/L_{\hat{r},t})\left(\frac{\partial Y_{\hat{r},t}}{\partial q_{\hat{r},t}} - \tau_{\hat{r},t}\right) = \mu_0 \tag{59}$$

[36] Wages are not equated across locations due to labor immobility.

Equating this condition to the first order (34) of the planner it follows directly, after making use also of (30), that if welfare weights are set so that $\upsilon(\hat{r}) = 1/\Lambda_{\hat{r}}^k$ and taxes according to

$$\tau_t^* \equiv \Lambda_t^m \quad \forall \hat{r} \tag{60}$$

competitive markets will implement the solution to planning problem.

From the results derived so far we see that the competitive equilibrium produces the same necessary optimality conditions as in the planning problem. Hence, the following proposition follows[37]:

Proposition 1. *The set of welfare weights* $\{v(\hat{r})\}_{\hat{r}}$ *that produce the Pareto efficient outcome corresponding to the competitive equilibrium with zero transfers must satisfy* $\upsilon(\hat{r}) = 1/\Lambda_{\hat{r}}^k$.

This is simply an application of the first theorem of welfare economics. Alternatively, denote λ_t^k as the Lagrangian multiplier of the planner's budget constraint. From the results derived so far we see that the price of the consumption good in competitive equilibrium must rise at the same rate as the multiplier of the planner's budget constraint i.e. $p_t^k/p_{t+1}^k = \lambda_t^k/\lambda_{t+1}^k$. Without loss of generality we can normalize consumer prices and the Lagrangian multiplier so that $p_0^k = \lambda_0^k$, which thus implies that equilibrium prices will correspond exactly to the Lagrangian multipliers of the planning problem. Hence, it follows directly that by setting $\upsilon(\hat{r}) = 1/\Lambda_{\hat{r}}^k$ the first order condition with respect to consumption in the planning problem matches exactly the first order condition of the representative consumer.[38] Finally, from the first order conditions of the goods producers it also follows that after inserting these into the consumers budget constraint the resulting expression satisfies the planner's budget constraint.

From the discussion above we have thus asserted that the competitive equilibrium described in Section 3.4 is indeed Pareto optimal given that welfare weights are defined appropriately. However this is not the end of the story, following the second theorem of welfare economics all other Pareto efficient allocations associated with different welfare weights might also be attained as a competitive equilibrium with transfers. All that is required is a set of transfers that adjust the multiplier $\Lambda_{\hat{r}}^k$ so that its reciprocal equals the weight $\upsilon(\hat{r})$ assigned to location $\hat{r}$. We summarize this in the following proposition which is a consequence of the second theorem of welfare economics.

Proposition 2. *For any arbitrary set of welfare weights* $\{v(\hat{r})\}_{\hat{r}}$ *the planning problem may be decentralized as a competitive equilibrium with transfers.*

This confirms within the context of our dynamic two-dimensional energy balance model, the static result of e.g. Chichilnisky and Heal (1994), Chichilnisky et al. (2000), Sandmo (2006), that if lump sum transfers between regions are possible, optimal Pigouvian taxes on the externality producing activity should be harmonized across countries. It also follows that if taxes are set according to (60), by the Second Welfare Theorem any Pareto optimal allocation defined for some set of welfare weights $\upsilon(\hat{r})$ may be attained, as a market equilibrium after the appropriate shuffling of endowments across locations have been made.[39]

3.6. *Optimal planning problem in a closed market economy*

Turn now to the *closed economy*. In this case each country's capital budget constraint must be independently satisfied so that $B_{\hat{r},t} = 0\ \forall \hat{r}, t$. The price of the consumption good is also assumed to be location dependent an thus denoted by $p_{\hat{r},t}^k$ which replaces p_t^k in the consumers budget constraint of Section 3.4. In this case each specific country is essentially a closed economy where the planner's role is only to determine the optimal tax rate at each location. As will be shown below, this scenario can lead to the emergence of a spatially differentiated optimal carbon tax. The appearance of this result is most easily shown by assuming that each location has its own deposit of fossil fuels $\mathcal{R}_0(\hat{r})$ where the extraction from this resource by the location dependent representative fossil fuel firm leads to location specific prices $p_{\hat{r},t}$ and extraction flows $q_{\hat{r},t}$ which must satisfy $\sum_{t=0}^{\infty} q_{\hat{r},t} = \mathcal{R}_0(\hat{r})$.[40] We thus assume that location $\hat{r}$ has full ownership of its respective fossil fuel firm and hence collects its full profits from extraction i.e. $p_{\hat{r},t} q_{\hat{r},t}$ replaces $s_{\hat{r}} p_t q_t$ in Eq. (47). This is not a specifically realistic case but it helps to bring out the forces that may generate spatially differentiated carbon taxes. The welfare maximization problem of

[37] We do not provide a formal proof of Propositions 1 and 2 which would also require a proof for the existence and uniqueness of the competitive and planning equilibrium. Here we only match FONC. A full proof lies beyond the scope of the present paper. A sketch of proof may be attained from the authors upon request.

[38] The first order conditions of the planning problem implies that $\upsilon(\hat{r}) U'(C_{\hat{r},t}/L_{\hat{r},t}) = \lambda_t^k$. Hence, it is clear that setting $\upsilon(\hat{r}) = 1/\Lambda_{\hat{r}}^k$ this t the competitive equilibrium (49) will be satisfied.

[39] In a complete treatment, transfers would also appear in the $I_{\hat{r},t}$ term, e.g. in (47). Ideally, for each $v(\hat{r})$ we would then denote the optimum allocation of consumption, capital, fossil fuels to each $(\hat{r}, t)$ by {*} say. Then we would exhibit a formula for each transfer $\mathrm{Tr}_{\hat{r},t}^*$ in terms of {*}, place this on the RHS of the income term (47) and show that facing this set of transfers, a competitive equilibrium would be produced that has the same allocations of capital, consumption, and fossil fuel usage as does {*}. See e.g. Kehoe et al. (1990) for an application within a standard macro setting without the extra complication of externality adjusted taxes.

[40] The competitive equilibrium is thus modified by replacing $\mathcal{R}_0$ with $\mathcal{R}_0(\hat{r})$ in (56) and p_t by $p_{\hat{r},t}$.

the planner can now be written as

$$\max \sum_{t=0}^{\infty} \beta^t \int \upsilon(\hat{r}) L_{\hat{r},t} U(C_{\hat{r},t}/L_{\hat{r},t})\, d\Phi\, dt \tag{61}$$

subject to (11), (9), and (19) and $\sum_{t=0}^{\infty} q_{\hat{r},t} = \mathcal{R}_0(\hat{r})$ with $B_{\hat{r},t} = 0\ \forall \hat{r}, t$. Each country is thus restrained by its own permanent income constraint. The first order necessary conditions w.r.t. $K_{\hat{r},t+1}$ gives us the standard Euler equation

$$U'\left(\frac{C_{\hat{r},t}}{L_{\hat{r},t}}\right) = \beta U'\left(\frac{C_{\hat{r},t+1}}{L_{\hat{r},t+1}}\right)\left(\frac{\partial Y_{\hat{r},t+1}}{\partial K_{\hat{r},t+1}} + (1-\delta)\right) \tag{62}$$

Note that this condition is equivalent to the one obtained by combining (51) and (53) from the market equilibrium which implies that given that taxes and transfers are administered by the market constrained planner this implies that each country is an independent market with its own interest rates and saving/investment decisions.

The first order necessary conditions w.r.t. $q_{\hat{r},t}$ is given by

$$\beta^t \upsilon(\hat{r}) U'\left(\frac{C_{\hat{r},t}}{L_{\hat{r},t}}\right)\frac{\partial Y_{\hat{r},t}}{\partial q_{\hat{r},t}} = -\sum_{k=2}^{\infty} \beta^{t+k} \int \nu(\hat{r}) U'\left(\frac{C_{\hat{r},t+k}}{L_{\hat{r},t+k}}\right)\frac{\partial Y_{\hat{r},t+k}}{\partial T_{\hat{r},t+k}}\left(\sum_{s=1}^{k-1} \frac{\partial T_{\hat{r},t+k}}{\partial M_{t+s}}\frac{\partial M_{t+s}}{\partial q_t}\right) d\Phi + \mu_0(\hat{r}) \tag{63}$$

where $\mu_0(\hat{r})$ denotes the Lagrangian multiplier for the resource constraint at location $\hat{r}$.[41] The interpretation of this expression is as in the previous section. The first order condition states that marginal social utility of using an extra unit of fossil fuel (left hand side) must equal the marginal social cost (right hand side) at each respective location. As in the case of the planning problem in Section 3.2 we can write this in terms of a location specific externality adjusted Hotelling rule

$$\frac{U'(C_{\hat{r},t}/L_{\hat{r},t})}{\beta U'(C_{\hat{r},t+1}/L_{\hat{r},t+1})} = \frac{\frac{\partial Y_{\hat{r},t+1}}{\partial q_{\hat{r},t+1}} - \Lambda^m_{\hat{r},t+1}}{\frac{\partial Y_{\hat{r},t}}{\partial q_{\hat{r},t}} - \Lambda^m_{\hat{r},t}} \tag{64}$$

where the marginal externality costs of fossil fuel use per location $\Lambda_{x,t}$ is given by

$$\Lambda^m_{\hat{r},t} \equiv -\frac{1}{\beta^t \upsilon(\hat{r}) U'(C_{\hat{r},t}/L_{\hat{r},t})}\sum_{k=2}^{\infty} \beta^{t+k} \int \nu(\hat{r}) U'\left(\frac{C_{\hat{r},t+k}}{L_{\hat{r},t+k}}\right)\frac{\partial Y_{\hat{r},t+k}}{\partial T_{\hat{r},t+k}}\left(\sum_{s=1}^{k-1} \frac{\partial T_{\hat{r},t+k}}{\partial M_{t+s}}\frac{\partial M_{t+s}}{\partial q_t}\right) d\Phi \tag{65}$$

To see that the outcome of the planning problem may also be achieved as a competitive market equilibrium we start by observing that the Euler equation (62) of the planning problem can also be achieved through the first order conditions (51) and (53) of the market equilibrium where $B_{\hat{r},t} = 0\ \forall \hat{r}, t$ by assumption. Next, by comparing the competitive market rule (57) after making resource multiplier location specific ($\mu(\hat{r})$), to the corresponding planning version (63) it follows that for the competitive market equilibrium to be Pareto optimal, taxes must be set according to

$$\tau_{\hat{r},t} = \Lambda^m_{\hat{r},t} \quad \forall \hat{r}, t \tag{66}$$

As with the *open economy* planning problem, the first welfare theorem requires setting welfare weights so that induced transfers are zero which implies that welfare weights must satisfy $\upsilon(\hat{r}) = 1/\Lambda^k_{\hat{r}}$.[42] For a different arbitrary set of welfare weights $\tilde{\upsilon}(\hat{r})$ the corresponding Pareto optimal allocation may however also be attained given the appropriate amount of transfers which adjusts the multiplier $\Lambda^k_{\hat{r}}$ of the consumer until $\tilde{\upsilon}(\hat{r}) = 1/\Lambda^k_{\hat{r}}$ holds. Once again, this is the second theorem of welfare economics at work.

As we have now shown that this market constrained planning problem is attainable as a competitive equilibrium with transfer, we now turn to studying how different welfare weights impact on optimal fossil fuel use and welfare across space. From Eq. (63) we see how the solution to this problem may generate spatially differentiated taxes. To see this we start by defining

$$\lambda^m_t \equiv -\sum_{k=2}^{\infty} \beta^{t+k} \int \nu(\hat{r}) U'\left(\frac{C_{\hat{r},t+k}}{L_{\hat{r},t+k}}\right)\frac{\partial Y_{\hat{r},t+k}}{\partial T_{\hat{r},t+k}}\left(\sum_{s=1}^{k-1} \frac{\partial T_{\hat{r},t+k}}{\partial M_{t+s}}\frac{\partial M_{t+s}}{\partial q_t}\right) d\Phi \tag{67}$$

Next, substituting the f.o.c. of the firm w.r.t. fossil fuel use, which for the *closed economy* problem is given by $\partial Y_{\hat{r},t}/\partial q_{\hat{r},t} = p_{\hat{r},t} + \tau_{\hat{r},t}$ into Eq. (63) we have

$$\frac{\mu_0(\hat{r}) + \lambda^m_t}{\beta^t \upsilon(\hat{r}) U'(C_{\hat{r},t}/L_{\hat{r},t})} = p_{\hat{r},t} + \tau_{\hat{r},t} \tag{68}$$

[41] Note that the integral over $\hat{r}$ in (63) appears since $q_{\hat{r},t}$ contributes equally to warming at each respective location regardless of where it has been emitted.

[42] As in Section 3.5, by inspection of the f.o.c. the price of the consumption good in competitive equilibrium must rise at the same rate as the multiplier of the planner's budget constraint i.e. $p^k_{\hat{r},t}/p^k_{\hat{r},t+1} = \lambda^k_{\hat{r},t}/\lambda^k_{\hat{r},t+1}$. Normalizing prices so that $p^c_{\hat{r},0} = \lambda^k_{\hat{r},0}$ thus implies that $p_{\hat{r},t} = \lambda^k_{\hat{r},t}$ and that equilibrium prices exactly equal the Lagrangian multiplier of the planner's problem.

where the right hand side can be interpreted as the *social price* of fossil fuel use payed by location x, i.e. the final price after the externality has been internalized. By this expression we see that it is only in special cases that the right hand side of the above expression will be equal across locations. This requires that $\mu_0(\hat{r}) = \mu_0(\hat{r}')$ and that the welfare weights are so-called Negishi weights defined as the inverse of marginal utility of consumption. Further, we can see that if $\mu_0(\hat{r}) = \mu_0(\hat{r}')$ and welfare weights instead are set equally and independent of locations, the location having the highest marginal utility of consumption should pay a lower social price of fossil fuel use. This will be the case when for example, the resource of x and the resource of $\hat{r}'$ are very large, i.e. the "Hotelling rent" is small in $\hat{r}, \hat{r}'$.[43] We formalize this in the following proposition

Proposition 3. *Assume that $\mu_0(\hat{r}) = \mu_0(\hat{r}')$ and $C_{\hat{r}',t}/L_{\hat{r}',t} < C_{\hat{r},t}/L_{\hat{r},t}$ and that welfare weights are set equal and independent of $\hat{r}$ at all locations ($\upsilon(\hat{r}) = \upsilon$). Then if the planner is unable to administer transfers, so that the weighted marginal utilities of consumption are equated across locations, then the location located at $\hat{r}'$ will pay a lower social price for fossil fuel use relative to a location $\hat{r}$.*

The result of (3) is most easily seen by taking the ratio between location $\hat{r}$ and the point ($\hat{r}'$) from (68). Under the assumptions of (3) we thus have

$$\frac{U'(C_{\hat{r}',t}/L_{\hat{r}',t})}{U'(C_{\hat{r},t}/L_{\hat{r},t})} = \frac{p_{\hat{r},t} + \tau_{\hat{r},t}}{p_{\hat{r}',t} + \tau_{\hat{r}',t}} \tag{69}$$

Hence, we see that by (3) if a location $\hat{r}'$ is expected to be poorer in the sense that $C_{\hat{r}',t}/L_{\hat{r}',t} < C_{\hat{r},t}/L_{\hat{r},t}$ this implies by the concavity of the utility function that $\hat{r}'$ should also pay a lower social price relative to other locations.

4. A numerical simulation of the closed economy model

In the previous sections we derived some general results regarding the optimal mitigation policies and their spatial and temporal profiles. Despite their simplistic appearance, calibrations of EBCMs that provide decent two dimensional representations of the climate system can still become quite involved. Typically, many papers using EBCMs rely on a minimum of two digit truncation schemes. Hyde et al. (1989) used a T11 type truncation scheme which involves solving 432 complex algebraic equations. When comparing a slightly generalized version of the EBCM used by North et al. (1983) to a full scale general circulation model they found that the predicted temperature changes of the EBCM in general agreed well with the general circulation model and that computational time was significantly lower.[44] However, when dealing with economic models involving maximization schemes, it is apparent that even small scale EBCMs will become difficult to handle due to the large number of state variables introduced even with low order truncation schemes.

In the current section we will proceed with a simple numerical exercise for the closed economy derived in Section 3.6 which we hope provides a good introduction to numerical problem at hand when using EBCMs in conjunction with economic modeling. We have chosen to model the closed economy since we believe it lies closest to the RICE model by Nordhaus and Boyer (2000), which inline with Section 3.6 also assumes that regions are economically autonomous and impacted heterogeneously by climate change. An essential difference between the RICE model and our model is the possibility for diffusion of heat across different regions which are an essential part of our EBCM framework. A simpler case to consider would be the open economy version of our model but with the trade-off of less realism due to the assumptions of free flows of resources across regions. Ideally an intermediate case would be preferable which we leave for the future work.[45]

The aim of this section is thus to show how heat transportation matters and also give a flavor of what quantitative results may be expected using more realistically calibrated models. It is thus worth stressing that this exercise is not be seen as a serious attempt to give precise answers, but rather as a finger exercise and a first attempt on capturing the value added of spatially dependent EBCMs as opposed to simpler representations of the climate economy interaction. With this in mind we will proceed the numerical exercise with some further simplifications in order to increase tractability and ease of exposition. As will be seen below, from these simplifications it will follow that location temperatures across the longitudinal dimension will be uniformly distributed which thus essentially transforms our two-dimensional model into the one-dimensional framework we considered in Brock et al. (2013a). Although, viewing latitudes as distinct economic zones is a wild abstraction from reality we believe this to be an important first step in developing these types of models for economic analysis. Future work in a full blown two-dimensional setting is thus an important step for future work on these types of problems.

The role of heat transportation can be captured in several ways within EBCM framework. The extension we include here is related to a phenomenon known as *polar amplification* which is a climate pattern implying a greater amount of warming

[43] Another example of when $\mu_0(\hat{r}) = \mu_0(\hat{r}')$ holds is when we as in Section 3.3.2 assume, a logarithmic utility function, Cobb–Douglas production and full capital depreciation then it is clear that we can attain a fixed rule for consumption $C_{\hat{r},t} = (1-\alpha_K\beta)Y_{\hat{r},t}$ and investment $K_{\hat{r},t+1} = \alpha_K\beta Y_{\hat{r},t}$ as we did in the *open economy* problem. Applying these rules to Eq. (63) implies that the left hand side depends on $\hat{r}$ only through the welfare weight $\upsilon(\hat{r})$. The solution to optimal fossil fuel is now completely characterized by the climate dynamics and fossil fuel constraint $\sum_{t=0}^{\infty} q_{\hat{r},t} = \mathcal{R}_0(\hat{r})$. Hence, by solving (63) for $q_{x,t}$ and substituting this into the fossil fuel constraint it is easy to show that $\mu_0(\hat{r}) = \mu_0(\hat{r}')$ holds whenever $\mathcal{R}_0(\hat{r})/(\upsilon(\hat{r})L_{\hat{r},0}) = \mathcal{R}_0(\hat{r}')/(\upsilon(\hat{r}')L_{\hat{r}',0})$.

[44] According to Hyde et al. (1989) their model could be solved in under 3 min on a standard desktop computer as opposed to the general circulation model which required about 70 h on a super computer.

[45] An intermediate case was would be to consider the case of costly transfers. We explore this to some extent within Brock et al. (2013a).

at high latitudes compared to the rest of the globe in response to an increase in global radiative forcing (i.e. global warming tends to be amplified at the poles). Recent progress in climate science has investigated the source of polar amplification (see e.g. Alexeev et al., 2005; Cai, 2005; Graversen et al., 2008; Graversen and Wang, 2009; Alexeev and Jackson, 2012). Two processes are believed to be the main cause; (i) changes in snow/sea ice, induces a positive surface albedo feedback, the less snow/ice, the more sunlight gets absorbed back thus creating the potential for warmer temperatures and further melting, (ii) due to changes in atmospheric heat transport mechanisms caused by a warming atmosphere. A warmer atmosphere holds more moisture and with an unchanged temperature gradient and eddy activity more heat will be transported polewards. The first process is standard and has been explored extensively in the EBCM literature. We have also explored this feedback in Brock et al. (forthcoming).

The second is more recent extension and did not receive much attention in earlier energy balance models but has been picked up recently as an alternative mechanism of polar amplification (Alexeev et al., 2005; Cai, 2005). We follow Alexeev et al. (2005) here and extend the original North et al. (1983) model to incorporate changes in atmospheric heat transport mechanisms by making the diffusion coefficient dependent on mean global temperature according to

$$D(\overline{T}) = D_0(1+\vartheta(\overline{T}-T_{pre})) \tag{70}$$

so that diffusion increases as the global mean temperature $\overline{T}$ rises above a reference temperature T_{pre} which we set equal to the preindustrial level. This causes polar amplification also in the absence of an ice-albedo feedback. The EBCM model can now be written as follows

$$\mathrm{C}(T_{\hat{r},t+1}-T_{\hat{r},t}) = QS(x)\alpha(x)-[A+BT_{\hat{r},t}]+h_t+\nabla D(\overline{T})\nabla T_{\hat{r},t} \tag{71}$$

The first thing to notice with Eq. (71) is the constant heat capacity C. A similar model featuring stochastic forcing was studied by North and Cahalan (1981). As will be seen in the calibration section below, this model has the property that if the carbon dioxide forcing is held constant the equilibrium temperature distribution will be uniform across longitudes. The constant heat capacity thus implies that the model will not be able to provide a reasonable representation of the actual longitudinal temperature distribution found in e.g. North et al. (1983). On the other hand, the model offers a great deal of analytical tractability since the solution to the partial differential equation is obtained as a set of linear and mutually independent ordinary differential equation (see Eq. (9)). Further, given that one has a good understanding of the model framework we pursue here it should be relatively straightforward for the numerically skilled economist to extend the current framework to solve more complex models with non-uniform longitudinal temperature distributions such as North et al. (1983), Hyde et al. (1989). This is something we leave for future research.

In order to make further progress with Eq. (71) using spherical harmonics, the next step is then to decide on a truncation order for (8). We adopt a low order triangular truncation scheme commonly known as T2. With this truncation level (8) can now be written as

$$T_{\hat{r},t} = T_{0,0,t}Y_{0,0}(\hat{r})+\sum_{m=-1}^{1}T_{1,m,t}Y_{1,m}(\hat{r})+\sum_{m=-2}^{2}T_{2,m,t}Y_{2,m}(\hat{r}) \tag{72}$$

Substituting (72) into (71) and integrating over the sphere we have the following system:

$$\mathrm{C}(T_{0,0,t+1}-T_{0,0,t}) = QH_{0,0}-\sqrt{4\pi}(A-h_t)-BT_{0,0,t} \tag{73a}$$

$$\mathrm{C}(T_{1,m,t+1}-T_{1,m,t}) = QH_{1,m}-(2D(\overline{T})+B)T_{1,m,t}, \quad \text{for } m=\{-1,0,1\} \tag{73b}$$

$$\mathrm{C}(T_{2,m,t+1}-T_{2,m,t}) = QH_{2,m}-(6D(\overline{T})+B)T_{2,m,t}, \quad \text{for } m=\{-2,-1,0,1,2\} \tag{73c}$$

Regardless of the T2 truncation scheme being the lowest possible order of truncation without completely fudging the spatial dimension, this still gives us a substantial system including in total nine differential modes.[46] An important direction for future research is to explore the robustness of any conclusions to the degree of approximation of the solution by number of modes. Souganidis (2013) has studied robustness for the one dimensional model of Brock et al. (2013a) and has found substantial differences when one moves from two modes to four or eight. Although theory suggests that the solution will become more accurate as more modes are added, it is beyond the scope of our paper to investigate this issue further.

To get further with this system we follow Nordhaus (2007), Golosov et al. (2011), and calibrate the model assuming fairly long time period intervals of ten years. This is not ideal for policy analysis. For example, Cai et al. (2012b) demonstrate that many substantive results depend critically on the time step and if one wants to know how carbon prices should react to business cycle shocks, the time period needs to be at most a year. However, ten year intervals makes assumptions such as, full capital depreciation used in Section 3 more realistic as opposed to shorter intervals. Further, given this rather lengthy time step this also allows us to make another simplifying assumption without violating first principles to much. In every time step, we assume that temperature dynamics equilibrate fast compared to carbon dioxide and capital dynamics so that the modes in (9) relax fast to their respective steady states in every time period. This implies that for every change in

[46] Recalling the definitions $H_{n,m} = \int Y^*_{n,m}(\hat{r})S(x)\alpha(x)\,d\Phi$ and $h_t = \xi\ln(M_t/M_{pre})$. Also note that $\nabla D(\overline{T})\nabla T_{\hat{r},t} = D(\overline{T})\nabla^2 T_{\hat{r},t}$ which implies that we are still dealing with a two-dimensional Laplacian with eigenvalues $n(n+1)$ and eigenfunctions given by the spherical harmonics.

radiative forcing due to carbon dioxide emissions we assume that a ten year time interval suffices for atmospheric temperatures to reach the new equilibrium. There has been a wide spread debate in the climate community regarding the response time of the climate system to a changes in radiative forcing (see e.g. Schwartz, 2007; Foster et al., 2008; Knutti et al., 2008; Scafetta, 2008; Schwartz, 2008). The response time depends on a wide variety of factors in particular on how efficiently heat perturbations are mixed into the deeper ocean. Ocean mixing is complex and not necessarily simulated well by climate models. Empirical data on ocean heat uptake are improving rapidly, but still suffer limitations (Hansen and Sato, 2011). Much of the difficulty is also due to the fact some systems respond quickly such as the atmosphere, while others such as the deep ocean respond very slowly. Held et al. (2010) distinguish between a fast and a slow component of global warming and estimate a relaxation or e-folding time for the fast component to a fixed doubling of CO_2 from pre-industrial levels to be less than 5 years while the slow component achieves very little even within the scope of hundreds of years due to the slow response of the deep ocean (see also e.g. Knutti et al., 2008).[47] Based on the above literature, we conclude that assuming that the climate system reaches equilibrium every ten years constitutes a decent approximation when it comes to atmospheric and upper ocean levels but is far off when accounting for the deep oceans, glaciers and ice caps. Hence, future work should preferably include also an ocean module capable capturing this slower response time (see e.g. Rose and Marshall, 2009).

Following this assumption, we can easily see that setting (73a)–(73c) equal to zero, immediately implies that all modes with $m \neq 0$ go to zero in steady state (i.e. $\lim_{t\to\infty} T_{n,m,t} = 0 \ \forall m \neq 0$). This reduces the system from nine to three equations. Second, a common assumption often made in the EBCM literature is to treat the northern and southern hemispheres as symmetrical meaning that the mean-annual temperature distribution is symmetrical around the equator (see e.g. North, 1975a,b; North et al., 1981). This assumption implies that the temperature function will be an even function in latitude (i.e. $T_{x,\phi,t} = T_{-x,\phi,t}$) which allows us to state the following proposition[48]:

Proposition 4. *Assume a mean-annual, North–South symmetric temperature distribution i.e.* $T_{x,\phi,t} = T_{-x,\phi,t}$, *and that the temperature function can be approximated by a Fourier-spherical harmonic series so that* $T_{x,\phi,t} = \sum_{n,m} T_n^m Y_n^m(x,\phi)$. *Then all temperature modes* T_n^m *of this approximation where,* $n-|m|$ *is an odd number, will be zero (i.e.* $T_n^m = 0$ *for all odd* $n-|m|$*).*

Proof. See appendix.

From the assumptions we have made so far the system (73) has thus been substantially reduced to the point where we are actually dealing with a problem which in steady state is similar to the model we derived in Brock et al. (2013a). This can be seen by substituting the remaining equilibrium modes $T_{0,0,t}$ and $T_{2,0,t}$ into (72) which gives us

$$\begin{aligned} T_{\hat{r},t} &= T_{0,0,t} Y_{0,0}(\hat{r}) + T_{2,0,t} Y_{2,0}(\hat{r}) \\ &= \frac{1}{B}\left(QH_{0,0} - \sqrt{4\pi}(A-h_t)\right) Y_{0,0}(\hat{r}) + \frac{QH_{2,0}}{(6D(\overline{T})+B)} Y_{2,0}(\hat{r}) \\ &= \frac{1}{B}\left(\frac{Q}{2}\int_{-1}^{1} S(x)\alpha(x)\,\mathrm{d}x - (A-h_t)\right) + \frac{5Q}{2}\frac{\int_{-1}^{1} S(x)\alpha(x)P_2(x)\,\mathrm{d}x}{(6D(\overline{T})+B)} P_2(x) \end{aligned} \tag{74}$$

As it turns out, given the assumption we made above, we arrive at almost exactly the same expression as in Eq. (17) of Brock et al. (2013a) with the exception that there we did not consider any polar amplification processes. This reveals the close correspondence between the solutions to the one- and two-dimensional problem. Notice also that (74) only depends on x. This follows since the zero modes allow us to completely integrate out the longitudinal dimension. Hence, Eq. (74) implies that the temperature is the same along all longitudinal coordinates for each respective latitude x.

4.1. Calibration

To calibrate the model we adopt benchmark estimates from the literature where available. For the energy balance parameters we use mainly estimates from Alexeev et al. (2005). Here, the meridional heat distribution $S(x)$ is expressed in terms of a two-mode Legendre polynomial as $S(x) = 1 + S_2 P_2(x)$ with $S_2 = -0.482$ and the co-albedo by $a(x) = 0.681 - 0.202 P_2(x)$. Note, how the choice of the normalization coefficient of the spherical harmonics (which for the zero mode was $1/\sqrt{4\pi}$) ensures also that the integral of mean annual solar distribution multiplied by one half equals unity i.e. we have $0.5\int_{-1}^{1} S(x)\,\mathrm{d}x = 1$ (see North et al., 1981 for a derivation). Other estimates are taken from Alexeev et al. (2005) including $Q = 340\ \mathrm{W/m^2}$, $D_0 = 0.445$, $\vartheta = 0.03$ and $B = 2\ \mathrm{W/m^2\,K}$ while T_{pre} is set to 14 °C.

Next, we calibrate the model to get the observed average estimate of present global temperature of 15 °C (see North and Stevens, 2006). Current estimates of global carbon dioxide are approaching 400 ppm which translates to approximately 848 GtC. For the pre-industrial concentration of carbon dioxide we use an estimate from Nordhaus (2007) and set $M_{pre} \approx 596$ GtC. For, ξ we set this to 5.35 W/m^2 (Myhre et al., 1998). We then adjust A so that the global mean temperature

[47] E-folding time or relaxation time refers to the time it takes for a perturbed system to reach a point where only a fraction $1/e$ of the initial difference between the initial value of the observable and the equilibrium value where $e \approx 2.7182$.

[48] Here, we have written out the components x and ϕ of the vector $\hat{r}$ explicitly i.e. $T_{\hat{r},t}$ and $T_{x,\phi,t}$ are notational equivalents.

equals 15 °C i.e.

$$\overline{T} = \frac{1}{B}\left(\frac{Q}{2}\int_{-1}^{1} S(x)\alpha(x,x_s)\,dx - A - \xi \ln\left(\frac{M_t}{M_{pre}}\right)\right) \approx 15\ ^\circ\mathrm{C} \tag{75}$$

given the above estimates we have that A equals 206 $\mathrm{W/m^2}$.[49]

Turning to the carbon dioxide dynamics two parameters need to be calibrated. First, the so called airborne fraction η which is the fraction of anthropogenic carbon dioxide emissions that remains in the atmosphere. We model this fraction as constant hence assuming that there is no trend in the biospheres and oceans ability to absorb human induced emissions. Based on the study by Knorr (2009) we set the airborne fraction $\eta = 0.43$.[50] Second, the parameter φcaptures the rate at which carbon is absorbed by the deep oceans. Archer (2005) claims that "…75% of an excess atmospheric carbon concentration has a mean lifetime of 300 year and the remaining 25% stays there forever". Although our simple carbon cycle is unable to account for the 25% always remaining in the atmosphere by setting m=0.05 this implies that after 300 years (30 periods) approximately 75% of the carbon dioxide has been removed. Finally, E_{land} is backed out from the relationship $M_{pre} = E_{land}/m$ which implies a value of 29.8.

For the economic parts of the model, we rely on estimates from Golosov et al. (2011) and Nordhaus (2007). As in Golosov et al. (2011) we assume a logarithmic utility function, Cobb–Douglas production and full depreciation of capital. These assumptions greatly simplify the numerical analysis since the dynamic investment problem of the model can be separated from decisions on inter temporal fossil fuel use as shown in Section 3.3.2.[51] For the Cobb–Douglas production function the factor shares and discount rate are $\alpha_K = 0.3$, $\alpha_q = 0.03$ and $\beta = 0.985^{10}$. Next, fossil fuel use in our model also requires an estimate of the current stock $\mathcal{R}_0$. We use the global reserve estimate of 5000 GtC from Rogner (1997) which also accounts for technical progress in extraction. This estimate consists of both coal and oil. It should be noted that the way we model fossil fuel production in this paper is simplistic in the sense that we ignore important extraction costs and refinement costs which are generally much higher for coal than oil. For a detailed model capturing the difference between coal and oil the reader is once again referred to Golosov et al. (2011). As we want to specifically focus on the spatial properties inherent in our model we have ignored this distinction in order to keep the model simple.[52]

Next, we turn to the parameters of the model that are distributed across space. This concerns the damage function, fossil fuel resources and welfare weights. Since, we are not attempting to give any precise answers in our model simulations, we will rely on rough functional approximations of these distributions which we will fit o the corresponding empirical distributions by simple ocular inspection. A distribution function that will do a decent job of approximating all these distributions, for the purpose of the present paper, is

$$f(x) = \frac{(1-x)^a(1+x)^b(c+dx^2)}{\int_{-1}^{1}(1-x)^a(1+x)^b(c+dx^2)\,\mathrm{d}x} \tag{76}$$

which can be fitted to our empirical distributions of interest by adjusting the parameters a, b, c and d.

Starting with the damage function, we assume an exponential form ($\Omega(T) = e^{-\gamma T}$) and set the damage parameter γ to 0.01 which provides a decent approximation to the quadratic damage function used in Nordhaus (2007) (e.g. a temperature increase of 4° corresponds to $\approx 5\%$ loss of output). However, since our model is spatial we should also consider how damages are distributed across space. Based, on work by e.g. Mendelsohn et al. (2006) it is evident that climate change is expected to be most severe in poor countries surrounding the equator with a skew towards southern latitudes. We approximate this distribution using (76) by setting the parameters $a = 4, b = 3, c = 1$ and d=0. The shape of this distribution is given in Fig. 1 by the solid line. Next, fossil fuel use in our model also requires an estimate of the current stock. We use the global reserve estimate of 5000 GtC of oil equivalent from Rogner (1997). The approximate distribution of this global estimate can then be found by inspecting the global distribution of these reserves. This reveals a clear northern skew in the distribution of reserves.[53]

We approximate this distribution by setting $a = 3$, $b = 6$, $c = 1$ and d=0 in (76) which can also be seen by the dash-dotted line in Fig. 1. Finally, we turn to the distribution of welfare weights. We have for sake of comparison with other modeling work chosen to adopt a Negishi approach and set these according to GDP per capita estimates across latitudes found in Kummu and Varis (2011).[54] However, we recognize the substantive critique of the Negishi approach to welfare weights (see e.g. Stanton, 2010), but it lies beyond the scope of the present paper to do a full analysis of the implications for climate policy.[55] Once again, we will approximate the actual distribution of weights using the distribution function in (76) where we set $a = 2$, $b = 2.2$, $c = 0.01$ and d=1 given by the dashed line in Fig. 1.

[49] In this calculation we assume a fixed value of 0.7 for the co-albedo.

[50] Although there exists several studies have reported an apparent increasing trend in the airborne fraction the study by Knorr (2009) claim that this trend is statistically insignificant.

[51] For a discussion regarding how well these assumptions fit historical economic data the reader is referred to the discussion in Hassler et al. (2012) and Golosov et al. (2011).

[52] Adding these distinctions at a later stage should however, not affect any of the qualitative results of this paper.

[53] An exception is Latin America which holds a significant chunk of world reserves.

[54] See Kummu and Varis (2011) for a nice exposition of the geographical distribution of a myriad of social indicators including GDP and population estimates across latitudes.

[55] For a treatment on inequality under climate change in the context of the RICE model see Dennig (2013).

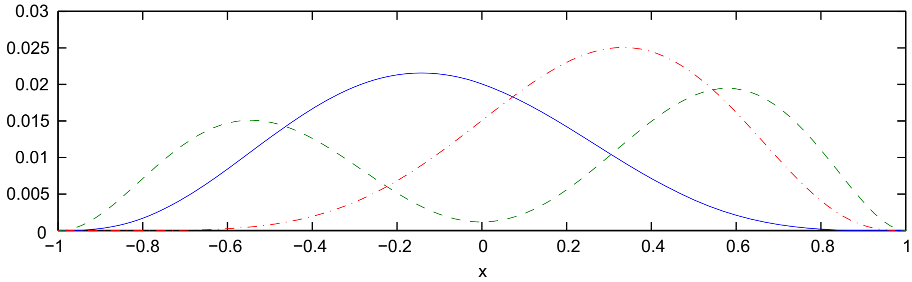

Fig. 1. Approximation of the empirical distribution functions using Eq. (76) for damages (solid line), fossil fuel reserves (dash-dot) and welfare weights (dashed).

4.2. Results

As we described above, the simplifications on behalf of the climate module, resulted in a one-dimensional problem along latitudes from the south to the north pole, where the sine of the latitude equals -1 a the south pole an 1 at the north pole. For the numerical computations the interval is discretized into S equally spaced grid points and the infinite time horizon is truncated to a sufficiently large time interval $[0, T]$. The final discretized problem which we solve is specified in Appendix A.3.

From the first order necessary optimality conditions for this problem we derive a set of equations describing a discrete dynamical system in discrete space. To determine the optimal solution converging to the long-run stationary distribution we use a continuation strategy applied to the boundary value problem of the discrete system. The general idea behind this approach is analogous to the time-continuous case, described in Grass (2012). The known stationary distribution at infinity is used to start a continuation, where successively the distributed parameters $R_{0,i}$, γ_i and ν_i are changed from initially equally distributed values to the values we are interested in.[56]

Fig. 2 presents the results for optimal fossil fuel use at each respective latitude and the resulting distributions of temperature and damages across latitudes, together with the accumulation of atmospheric CO_2. The simulations were done for a total of 41 discretization points (latitudes) lying in the interval $[-1, 1]$ were the graphs in Fig. 2 show also interpolated results in between these points. Here, Negishi type weights are used where the distribution function for these weights are shown by the dashed line in Fig. 1. Here, temperatures are given as anomalies or departures from a baseline point defined as the pre-industrial temperature distribution. The upper left hand graph shows how the distribution of these temperature anomalies evolves over time. From the graph we see how the endogenously determined diffusion coefficient $D(\overline{T})$, due to Alexeev et al. (2005) extension, causes an increase in the diffusion of temperature across space as a result of an increase in the global mean temperature due to atmospheric carbon dioxide emissions. In particular, one notices by inspection of the graph why this process is called *polar amplification*. The reason is that the temperature increase is the largest in the polar regions, reaching a max somewhere between 100 and 150 years down the line, i.e. temperature anomalies are particularly amplified at the poles. The upper right hand graph shows the corresponding fossil fuel use per latitude. From the intense dark red color around 0.5 this indicates that the northern regions initially will be using the most fossil fuels. This is of course due to the assumption we made earlier which is depicted by the dash-dot line in Fig. 1 and reveals a northern skew in the distribution of fossil fuels. Polar regions which received a small chunk of initial fossil fuel reserves are thus also relatively smaller users of fossil fuels due to the closed economy, no transfer/trade, assumption we are analyzing. Next, the lower left hand graph displays damages as a percent of output. Here, we made the assumption, that impacts from a given increase in temperature are more severe in regions surrounding the equator with a slight skew towards southern latitudes. This can be seen by the solid line in Fig. 1. The implications of these assumptions become clear by inspection of Fig. 2 where we see that the bulk of actual damages lie in the regions where damage intensity is assumed to be the largest as can be seen by comparing with Fig. 1. In comparison to the equator, polar regions are thus affected less by a given amount of global warming despite the fact that the relative increase in temperature is larger in these regions. This would of course be different had we assumed a uniform distribution of the damage parameter. In this case damages would have been more severe in the polar regions due the polar amplification effect resulting from the temperature dependent diffusion coefficient. Finally, the bottom rightmost graph shows the temporal distribution of atmospheric carbon dioxide. As can be seen the atmospheric CO_2 stock increases relatively quickly but due to the finite stock of fossil fuels and removal rate of atmospheric carbon dioxide it then levels out to pre-industrial levels. Evidence from climate scientists suggests that this is a very unlikely scenario. Most studies indicate that carbon dioxide will remain in the atmosphere much longer (see e.g. Archer, 2005; Solomon et al., 2009). However, as we mentioned several times already, the purpose of this paper is only to be suggestive of how the EBCM framework can be used to answer questions of relevance to economist's working on climate-economy models and hence open the door for future work where more careful attention is paid to model specifications and calibration issues.

[56] For the actual computations we used a slightly modified version of MATCONT, which is a MATLAB package providing algorithms for pseudo-arclength continuation.

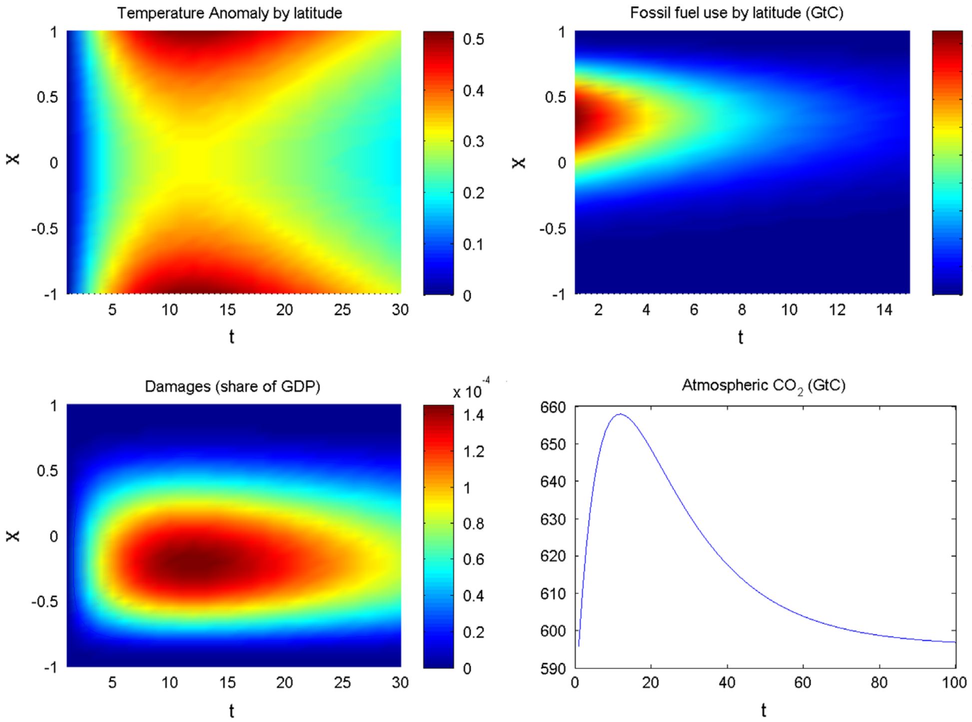

Fig. 2. Distributions of temperature, fossil fuel use and damages across latitudes and time and atmospheric CO_2 across time. *t* denotes time periods where one time period is equal to 10 years. *x* denotes the sine of the latitude.

5. Concluding remarks

In this paper we have developed a model of climate change consisting of a two-dimensional energy balance climate model which we coupled to a model of economic growth. We believe that modeling heat transport in the coupled model is the main contribution of our paper since it allows, for the first time, as far as we know, the derivation of latitude dependent temperature and dam- age functions, as well as optimal mitigation policies, in the form of optimal carbon taxes, which are all determined endogenously through the interaction of climate spatiotemporal dynamics with optimizing forward looking economic agents.

We derive Pareto optimal solutions for a social planner who seeks to implement optimal allocations with taxes on fossil fuels and we show the links between welfare weights and international transfers across locations and the spatial structure of optimal taxes. Our results suggest when per capita consumption across latitudes can be adjusted through costless transfers for any set of non-negative welfare weights, so that marginal valuations across latitudes are equated, or alternatively that transfers are zero due to Negishi weighting, then optimal carbon taxes will be spatially homogeneous. On the other hand when marginal valuations across latitudes are not equated, due to institutional/political constraints, optimal carbon taxes will instead be spatially differentiated. We show that if economies are autarkic and the planner is not constrained by Negishi weighting, then taxes on fossil fuels could be lower in relatively poor geographical areas. The degree of geographical tax differentiation will depend on the heat transport across latitudes. Without appropriate implementation of international transfers, and without Negishi weights that keep the existing international distribution invariant, carbon taxes will be latitude specific and their sizes will depend on the heat transfer across locations. We also show how to derive numerical results for the optimization problem of the unconstrained planning problem when economies are autarkic. These results show how the optimization model generates surface plots of temperature and damages across latitudes and over time and how these depend on the diffusion of temperature.

The climate module of our model is sometimes referred to as a surface EBCM where the impact of oceans is reflected in the carbon decay parameter *m*, with no further modeling of the deep ocean component is undertaken. Further extensions of our simple one-dimensional model to richer climate models (e.g. Kim and North, 1992; Wu and North, 2007) with a ocean and with simple atmospheric layers added and where tipping phenomena are possible may help to understand results like those of Challenor et al. (2006), which found higher probabilities of extreme climate change than expected. Challenor et al. (2006) suggest several reasons for their findings including “The most probable reason for this is the simplicity of the climate model, but the possibility exists that we might be at greater risk than we believed”. We emphasize that we are still doing what economists call a “finger exercise” in this paper where one deliberately posits an over simplified “cartoon” model in order to illustrate forces that shape, for example, an object of interest like, a socially optimal fossil fuel tax structure over time and space that might be somewhat robust to introduction of more realism into the toy model. For example, we believe that the interaction of spatial heat transport phenomena and difficulties in implementing income transfers (or their equivalent, e.g. allocations of tradable carbon permits) will play an important role in determining the shape of the socially optimal tax schedule over different parts of space in more complex and more realistic models. Our simple model is useful in making this type of point under institutions where income transfers are possible and where they are politically infeasible, i.e. essentially impossible.

The two-dimensional model allows the exploration of issues which cannot be fully analyzed in conventional zero-dimensional models. In particular two-dimensional models with spatially dependent co-albedo allow the introduction of latitude dependent damage reservoirs like endogenous ice-lines and permafrost. Since reservoir damages are expected to arrive relatively early and diminish in the distant future, because the reservoir will be exhausted, the temporal profile of the policy ramp could be declining or U-shaped.[57] A U-shaped policy ramp could be explained by the fact that high initial damages due to the damage reservoirs will start declining as the reservoir is exhausted, giving rice to a declining policy ramp, damages from the increase of the overall temperature will dominate causing the policy ramp to become increasing. Some of these aspects were explored in Brock et al. (forthcoming) in the context of a one-dimensional EBCM.

However, we believe that some of the most important areas for future research are the extensions of our framework to precipitation fields as well as temperature fields and to robust control. Our two dimensional energy balance model has been deliberately constructed to allow easy integration with the combined two dimensional energy balance moisture balance model of Fanning and Weaver (1996). This is so because our series expansion methods to approximate solutions to the model apply also to Fanning and Weaver (1996) model. One of the main gaps in received work by economists in the climate area is neglect of the precipitation and evaporation part of climate dynamics as well as ignoring transport phenomena in the dynamics of the temperature field across the globe. Many important sources of damages are more related to precipitation and evaporation than to temperature (e.g. Elliott and Fullerton, 2013 damages to agriculture).

Our two dimensional framework allows a direct extension to robust control which, in turn leads to identification of potential *hot spots* at various locations on the globe. We have already worked out some of the appropriate conceptualization of different kinds of *hot spots* and we have worked out quite a bit of the analytics (Brock et al., 2012, 2013b). It is essential that our approach be extended to incorporate model uncertainty and, hence, some form of robust control to take into account deep uncertainty in the dynamics of damages, climate dynamics, and especially economic dynamics where long term prediction is difficult due to the difficulties in predicting long term socio-economic response to climate change. So this is an extension we will undertake in the very near future. Incorporation of robust control into discussions of geoengineering (Manousi and Xepapadeas, 2013) is necessary in order to take into account deep uncertainties about potential effects on temperature and precipitation fields. Climate scientists have expressed serious concerns in this direction (Bala, 2012; Bala et al., 2008, 2011; Tuana et al., 2012; Goes et al., 2011). We share their concerns. Our framework seems like a useful platform for a robust control analysis of geoengineering, especially if our energy balance model is extended to an energy balance-moisture balance model like that of Fanning and Weaver (1996). We also need to extend our work to deal with issues raised by potential threshold effects and other concerns raised by climate scientists that must be addressed in formulation of robust climate policies (Keller et al., 2008; Hall et al., 2012). Another important direction for future research is extending our work to include more modeling of basic economic phenomena that we have left out in order to focus more on spatial climate dynamics. Desmet and Rossi-Hansberg (2012) is a recent paper that has costly trade, diffusion of technology, modeling of innovations, migration of labor, that we lack. Although they work on a hemisphere with latitude belts, they do not have heat transport climate dynamics like we do. They have a reduced form carbon cycle that includes the upper and lower ocean which we do not have. However we can add explicit coupling of spatial models of upper ocean and lower ocean to the dynamics of the atmospheric temperature field as in Rose and Marshall (2009) and our analytical methods will still apply. Desmet and Rossi-Hansberg (2012) are not able to address the potential impact of human activities on spatial phenomena such as amplification of the dynamics of the temperature field at high latitudes (Alexeev et al., 2005; Alexeev and Jackson, 2012) much less the potential impacts on spatial dynamics of precipitation and evaporation fields in models like (Fanning and Weaver, 1996; Weaver et al., 2001). But their treatment of the economic side is more thorough than ours.

Our solution methods for the climate module using spherical harmonics and our closed form solutions for the economic module (under the same set of sufficient conditions for existence of a closed form solution) carry right over to Fanning and Weaver (1996) energy balance moisture balance model as well.

Explicit modeling of the spatial dynamics of components of climate as we are doing combined with the explicit modeling of important economic components as in Desmet and Rossi-Hansberg (2012), more sectors as in Engström (2012), Hassler et al. (2012), is urgently needed. This kind of work will allow us to do a better job of accounting for implementation issues involving carbon pricing, for example, the very important issue of potential carbon leakage (Elliott and Fullerton, 2013).

Acknowledgments

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[57] Judd and Lontzek (2011) have formulated a dynamic stochastic version of DICE, the DSICE—which includes stochastic tipping points possibilities. They show that this complexity affects the optimal policy results in comparison to RICE.

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Appendix A

A.1. Static efficiency

Factor allocations Start by defining $\Psi_{\hat{r}}(T_{\hat{r},t}) \equiv \mathcal{A}_{\hat{r},0} L_{\hat{r},0}^{\alpha_L} \Omega(T_{\hat{r},t})$, $\mathcal{Y}_t \equiv (an^{\alpha_L})^{t-1} K_t^{\alpha_K} q_t^{\alpha_q}$, $S_{\hat{r},t}^K \equiv K_{\hat{r},t}/K_t$ and $S_{\hat{r},t}^q \equiv q_{\hat{r},t}/q_t$ so that

$$Y_{\hat{r},t} = (an^{\alpha_L})^{t-1} \mathcal{A}_{\hat{r},0} L_{\hat{r},0}^{\alpha_L} \Omega(T_{\hat{r},t}) K_{\hat{r},t}^{\alpha_K} q_{\hat{r},t}^{\alpha_q} = \mathcal{Y}_t \Psi_{\hat{r}}(T_{\hat{r},t})(S_{\hat{r},t}^K)^{\alpha_K}(S_{\hat{r},t}^q)^{\alpha_q}$$

The Lagrangian can then be written as

$$\mathcal{L} = \mathcal{Y}_t \int (\Psi_{\hat{r}}(T_{\hat{r},t})(S_{\hat{r},t}^K)^{\alpha_K}(S_{\hat{r},t}^q)^{\alpha_q})\, \mathrm{d}\Phi + \lambda_K \left(1 - \int S_{\hat{r},t}^K \,\mathrm{d}\Phi\right) + \lambda_q \left(1 - \int S_{\hat{r},t}^q \,\mathrm{d}\Phi\right)$$

The first order conditions are

$$\alpha_K \mathcal{Y}_t \Psi_{\hat{r}}(T_{\hat{r},t})(S_{\hat{r},t}^K)^{\alpha_K - 1}(S_{\hat{r},t}^q)^{\alpha_q} = \lambda_K \tag{A.1}$$

$$\alpha_q \mathcal{Y}_t \Psi_{\hat{r}}(T_{\hat{r},t})(S_{\hat{r},t}^K)^{\alpha_K}(S_{\hat{r},t}^q)^{\alpha_q - 1} = \lambda_q \tag{A.2}$$

Combining the first order conditions we can after some algebra solve for $S_{\hat{r},t}^K$ and $S_{\hat{r},t}^K$

$$S_{\hat{r},t}^K = \frac{\lambda_K}{\alpha_K \mathcal{Y}_t \Psi_{\hat{r}}(T_{\hat{r},t})}^{(\alpha_q - 1)/\alpha_L} \frac{\lambda_q}{\alpha_q \mathcal{Y}_t \Psi_{\hat{r}}(T_{\hat{r},t})}^{-\alpha_q/\alpha_L}$$

$$S_{\hat{r},t}^q = \frac{\lambda_q}{\alpha_q \mathcal{Y}_t \Psi_{\hat{r}}(T_{\hat{r},t})}^{(\alpha_K - 1)/\alpha_L} \frac{\lambda_K}{\alpha_K \mathcal{Y}_t \Psi_{\hat{r}}(T_{\hat{r},t})}^{-\alpha_K/\alpha_L}$$

Next, substitute these expressions into their respective constraints we can write

$$1 = \int S_{\hat{r},t}^K \,\mathrm{d}\Phi = \frac{\lambda_K}{\alpha_K \mathcal{Y}_t}^{(\alpha_q - 1)/\alpha_L} \frac{\lambda_q}{\alpha_q \mathcal{Y}_t}^{-\alpha_q/\alpha_L} \int \Psi_{\hat{r}}(T_{\hat{r},t})^{1/\alpha_L} \,\mathrm{d}\Phi$$

$$1 = \int S_{\hat{r},t}^q \,\mathrm{d}\Phi = \frac{\lambda_q}{\alpha_q \mathcal{Y}_t}^{(\alpha_K - 1)/\alpha_L} \frac{\lambda_K}{\alpha_K \mathcal{Y}_t}^{-\alpha_K/\alpha_L} \int \Psi_{\hat{r}}(T_{\hat{r},t})^{1/\alpha_L} \,\mathrm{d}\Phi$$

after substituting back in the conditions (A.(1) and A.2) into these expressions we have that

$$S_{\hat{r},t}^K = S_{\hat{r},t}^q = \Psi_{\hat{r}}(T_{\hat{r},t})^{1/\alpha_L} \Big/ \int \Psi_{\hat{r}}(T_{\hat{r},t})^{1/\alpha_L} \,\mathrm{d}\Phi$$

Next, after substituting these optimality shares back into the objective function it follows that:

$$\mathbb{Y}\left(K_t, q_t, \{T_{\hat{r},t}\}_{\hat{r}=-1}^{\hat{r}=1}, t\right) = \mathcal{Y}_t \int \Psi_{\hat{r}}(T_{\hat{r},t}) \frac{\Psi_{\hat{r}}(T_{\hat{r},t})^{1/\alpha_L}}{\int_{\hat{r}} \Psi_{\hat{r}}(T_{\hat{r},t})^{1/\alpha_L} \,\mathrm{d}\Phi}^{\alpha_K + \alpha_q} \mathrm{d}\Phi = (an^{\alpha_L})^{t-1} K_t^{\alpha_K} q_t^{\alpha_q} \left(\int (\mathcal{A}_{\hat{r},0} L_{\hat{r},0}^{\alpha_L} \Omega(T_{\hat{r},t}))^{1/\alpha_L} \,\mathrm{d}\Phi \right)^{\alpha_L}$$

Consumption. The Lagrangian is given by

$$\mathcal{L} = \int \upsilon(\hat{r}) L_{\hat{r},t} U(C_{\hat{r},t}/L_{\hat{r},t}) \,\mathrm{d}\Phi + \mu_C \left(C_t - \int C_{\hat{r}} \,\mathrm{d}\Phi \right)$$

The first order condition implies

$$\upsilon(\hat{r}) U'(C_{\hat{r},t}/L_{\hat{r},t}) = \mu_C \quad \forall \hat{r}$$

given the utility function defined in (17) with elasticity of marginal utility given by $-\nu$ we have that

$$C_t=\int\left(\frac{\mu_C L_{\hat{r},t}^{-\nu}}{\upsilon(\hat{r})}\right)^{-1/\nu} \mathrm{d}\Phi=\mu_C^{-1/\nu}\int \upsilon(\hat{r})^{1/\nu}L_{\hat{r},t}\,\mathrm{d}\Phi$$

changing the integration variables to $\hat{r}'$ and substituting μ_C for the first order condition we have

$$C_t=\upsilon(\hat{r})^{-1/\nu}(C_{\hat{r},t}/L_{\hat{r},t})\int \upsilon(\hat{r}')^{1/\nu}L_{\hat{r}',t}\,\mathrm{d}\Phi'$$

expression (29) follows directly and the maximized welfare function can thus be written as

$$\mathbb{U}(C_t)=\int \upsilon(\hat{r})L_{\hat{r},t}\frac{(\upsilon(\hat{r})^{1/\nu}(C_t/L_t))^{1-\nu}-1}{1-\nu}\,\mathrm{d}\Phi=L_t\frac{(C_t/L_t)^{1-\nu}}{1-\nu}-\int\frac{\upsilon(\hat{r})L_{\hat{r},t}}{1-\nu}\,\mathrm{d}\Phi \tag{A.3}$$

A.2. Proof of Proposition 4

Proof. The coefficients of the Fourier spherical harmonic series can be found by integration, using the orthogonality (7) of the spherical harmonics. For our temperature function they are given by $T_n^m=\int_0^{2\pi}\int_{-1}^{1} T(x,p)Y_n^{m*}(x,\phi)\,\mathrm{d}x\,\mathrm{d}\phi$. Since $T(x,\phi)$ is an even function in x and by the properties of the associated Legendre polynomials $P_n^m(x)$ is an odd function when $(n-|m|)$ is odd. This further implies that the product $T(x,\phi)P_n^m(x)$ will also be an odd function whenever $(n-|m|)$ is odd. For any odd function $f(x)$ it can be shown that $\int_{-a}^{a} f(x)\,dx=0$. Hence, since $Y_n^m(x,\phi)$ is separable in x and ϕ, this will thus imply that $T_n^m=0$ as long as $n-|m|$ is an odd number. □

A.3. Details of numerical simulations

Define $H_0 := \frac{1}{2}\int_{-1}^{1} S(x)\alpha(x)\,\mathrm{d}x$ and $H_2 := \frac{5}{2}\int_{-1}^{1} S(x)\alpha(x)P_2(x)\,\mathrm{d}x$ with $S(x)=1+S_2P_2(x)$ and $S_2=-0.482$ and the co-albedo by $a(x)=0.681-0.202P_2(x)$ where $P_2(x)=(3x^2-1)/2$ and $P_2(i):=P_2(x_i)$, with $x_i=2i/n-1$, $i=0,\dots,n$. Note also that since $\overline{T}:=T_{0,0,t}/\sqrt{4\pi}$ we may refer to $T_{pre}:=T_{0,0,0}/\sqrt{4\pi}$ as the pre-industrial. Hence, it follows by the definition of the zero mode that the global mean temperature anomaly becomes $\overline{T}-T_{pre}=1/B\xi\ln(M_t/M_{pre})$ an we can thus write the diffusion coefficient as a function of M_t i.e. $D(M_t)=D_0(1+\vartheta(\overline{T}-T_{pre}))=D_0(1+(\vartheta/B)\xi\ln(M_t/M_{pre}))$. Also, we denote $\tilde{\upsilon}$ and $\tilde{\beta}$ as the population adjusted welfare weights and discount rate respectively. The Lagrangian of the numerical problem can thus be written as

$$\begin{aligned}\mathcal{L}=\sum_{t=0}^{\infty}\tilde{\beta}^t&\Bigg\{\sum_i\tilde{\upsilon}(i)\ln(\Omega(T_{i,t})\mathcal{A}_{i,t}K_{i,t}^{\alpha_K}L_{i,t}^{\alpha_L}q_{i,t}^{\alpha_q}-K_{i,t+1})\\&+\lambda_t^m\left[\eta\sum_i q_{i,t}+(1-m)M_t+mM_{pre}-M_{t+1}\right]\Bigg\}\\&+\sum_i\mu_{0i}\left[\mathcal{R}_{0i}-\sum_{t=0}^{\infty}q_{i,t}\right]\\ \text{with}\quad T_{i,t}&:=\frac{1}{B}\left(QH_0-A+\xi\ln\left(\frac{M_t}{M_{pre}}\right)\right)+\frac{QH_2}{(6D(M_t)+B)}P_2(i)\end{aligned} \tag{A.4}$$

The first order necessary conditions become

$$\mathcal{L}_{K_{i,t+1}}=\tilde{\beta}^t\tilde{\upsilon}(i)\frac{1}{C_{i,t}}+\tilde{\beta}^{t+1}\tilde{\upsilon}(i)\frac{1}{C_{i,t+1}}\alpha_K\Omega(T_{i,t+1})\mathcal{A}_{i,t+1}K_{i,t+1}^{\alpha_K-1}L_{i,t+1}^{\alpha_L}q_{i,t+1}^{\alpha_q}=0$$

$$\begin{aligned}\mathcal{L}_{M_{t+1}}=\tilde{\beta}^{t+1}\sum_i\tilde{\upsilon}(i)\frac{1}{C_{i,t+1}}\Omega'(T_{i,t+1})\mathcal{A}_{i,t+1}K_{i,t+1}^{\alpha_K}L_{i,t+1}^{\alpha_L}q_{i,t+1}^{\alpha_q}\times\left(\frac{\xi}{B}\frac{1}{M_{t+1}}+\frac{QH_2P_2(i)}{(6D(M_{t+1})+B)^2}\frac{D_0\vartheta\xi}{B\sqrt{4\pi}}\frac{1}{M_{t+1}}\right)\\-\tilde{\beta}^t\lambda_t^m+\tilde{\beta}^{t+1}\lambda_{t+1}^m(1-m)=0\end{aligned}$$

$$\mathcal{L}_{q_{i,t}}=\tilde{\beta}^t\tilde{\upsilon}(i)\frac{1}{C_{i,t}}\alpha_q\Omega(T_{i,t})\mathcal{A}_{i,t}K_{i,t}^{\alpha_K}L_{i,t}^{\alpha_L}q_{i,t}^{\alpha_q-1}+\tilde{\beta}^t\eta\lambda_t^m-\mu_{0i}=0$$

After using applying the same methods as used in the previous sections we can thus state the complete system which can be solved numerically for $i\in\{1,2,\dots n\}$ regions as

$$\lambda_t^m=\tilde{\beta}\lambda_{t+1}^m(1-m)-\tilde{\beta}\sum_i\tilde{\upsilon}(i)\gamma_i\frac{1}{1-\tilde{\beta}\alpha_K}\left(\frac{\xi}{B}\frac{1}{M_{t+1}}+\frac{QH_2}{(6D(M_{t+1})+B)^2}\frac{D_0\vartheta\xi}{B\sqrt{4\pi}}\frac{1}{M_{t+1}}P_2(i)\right)$$

$$\tilde{\upsilon}(i)L_{i,t}\frac{1}{1-\tilde{\beta}\alpha_K}\alpha_q\frac{1}{q_{i,t}}+\eta\lambda_t^m=\tilde{\beta}\tilde{\upsilon}(i)\frac{1}{1-\tilde{\beta}\alpha_K}\alpha_q\frac{1}{q_{i,t+1}}+\tilde{\beta}\eta\lambda_{t+1}^m$$

$$M_{t+1}=\eta\sum_i q_{i,t}+(1-m)M_t+mM_{pre}$$

$$\mathcal{R}_{0i}=\sum_{t=0}^{\infty}q_{i,t}$$

List of parameter values[58]

$\tilde{\beta}$	α_K	α_q	m	M_{pre}	B	A	Q	D_0	ϑ	T_{pre}	ξ	η
0.985^{10}	0.3	0.03	0.05	596	2	206	340	0.445	0.03	14	5.35	0.43

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[58] See Section 4.1 for $\mathcal{R}_{0i}, \tilde{\upsilon}(i)$ and γ_i. Code is available upon request from the authors.

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